Chapter 3: Report of the Task Group on Conceptual Knowledge and Skills

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Abbreviations

ACT	American College Testing
ADP	American Diploma Project
CLE	course-level expectations
LSAY	High School and Beyond, Longitudinal Survey of American Youth
NAEP	National Assessment of Educational Progress
NCTM	National Council of Teachers of Mathematics
NELS	National Education Longitudinal Study
STPI	Institute for Defense Analyses Science and Technology Policy Institute
SAT	Scholastic Achievement Test
SES	socioeconomic status
TIMSS	Trends in International Math and Science Study

Executive Summary

Introduction

The National Mathematics Advisory Panel was asked to make recommendations on "the critical skills and skill progressions for students to acquire competence in algebra and readiness for higher-level mathematics." To address this particular charge, the Panel established a Task Group on Conceptual Knowledge and Skills (CKS). To guide its inquiry, deliberations, and recommendations, CKS formulated three major questions:

1) What are the major topics of school algebra?
2) What are the essential mathematical concepts and skills that lead to success in Algebra and that should be learned as preparation for Algebra?[1]
3) Does the sequence of topics prior to algebra course work or for algebra course work affect achievement in Algebra?

Methodology

The Panel was charged with determining how to use "the results of research relating to proven-effective and evidence-based mathematics instruction" and making recommendations "based on the best available scientific evidence." The Panel contracted with Abt Associates Inc. to survey the research literature for studies that addressed each task group's major questions and met standards of methodological quality.

The Task Group's literature review yielded some peer-reviewed and published studies that met standards of methodological quality and were relevant to the work of this Task Group, especially with respect to its third question. However, because of the small number of such studies, the Task Group decided to include reports that presented the best available evidence on the topic of the conceptual knowledge and skills needed for success in Algebra. Thus it supplemented the literature review with reports by national organizations and government agencies, and with analyses and comparisons of state curriculum frameworks and school textbooks developed for the Task Group by the Institute for Defense Analyses Science and Technology Policy Institute. Where there may be differences among these reports, studies, or analyses, the differences are so noted. The Task Group's recommendations on matters of definition and mathematical content were also guided by professional judgment.

[1] Algebra will be capitalized when it is referred to as a course.

Results and Conclusions

The Major Topics in School Algebra

The Major Topics in School Algebra that were developed by the Task Group on Conceptual Knowledge and Skills are shown in this section. The teaching of Algebra, like the teaching of all of school mathematics, must ensure that students are proficient in computational procedures, can reason logically and clearly, and can formulate and solve problems. For this reason, the topics listed below should not be regarded as a sequence of disjointed items, simply to be committed to memory. On the contrary, teachers and textbook writers should emphasize the connections as well as the logical progression among these topics. The topics comprise both core and foundational elements of school algebra—those elements needed for study of school algebra itself and those elements needed for study of more advanced mathematics courses. The total amount of time spent on covering them in single-subject courses is normally about 2 years, although algebra content may be and is often structured in other ways in the secondary grades. What is usually called Algebra I would, in most cases, cover the topics in Symbols and Expressions, and Linear Equations, and at least the first two topics in Quadratic Equations. The typical Algebra II course would cover the other topics, although the last topic in Functions (Fitting Simple Mathematical Models to Data), the last two topics in Algebra of Polynomials (Binomial Coefficients and the Binomial Theorem), and Combinatorics and Finite Probability are sometimes left out and then included in a precalculus course. It should be stressed that this list of topics reflects professional judgment as well as a review of other sources.

Symbols and Expressions
- Polynomial expressions
- Rational expressions
- Arithmetic and finite geometric series

Linear Equations
- Real numbers as points on the number line
- Linear equations and their graphs
- Solving problems with linear equations
- Linear inequalities and their graphs
- Graphing and solving systems of simultaneous linear equations

Quadratic Equations
- Factors and factoring of quadratic polynomials with integer coefficients
- Completing the square in quadratic expressions
- Quadratic formula and factoring of general quadratic polynomials
- Using the quadratic formula to solve equations

Functions

- Linear functions
- Quadratic functions—word problems involving quadratic functions
- Graphs of quadratic functions and completing the square
- Polynomial functions (including graphs of basic functions)
- Simple nonlinear functions (e.g., square and cube root functions; absolute value; rational functions; step functions)
- Rational exponents, radical expressions, and exponential functions
- Logarithmic functions
- Trigonometric functions
- Fitting simple mathematical models to data

Algebra of Polynomials

- Roots and factorization of polynomials
- Complex numbers and operations
- Fundamental theorem of algebra
- Binomial coefficients (and Pascal's Triangle)
- Mathematical induction and the binomial theorem

Combinatorics and Finite Probability

- Combinations and permutations, as applications of the binomial theorem and Pascal's Triangle

Critical Foundations of Algebra

The Task Group also presents three clusters of concepts and skills that it considers foundational for formal algebra course work:

1) Fluency With Whole Numbers,
2) Fluency With Fractions, and
3) Particular Aspects of Geometry and Measurement.

To prepare students for Algebra, the curriculum must simultaneously develop conceptual understanding, computational fluency, and problem-solving skills. These three aspects of learning are mutually reinforcing and should not be seen as competing for class time. The Critical Foundations identified and discussed below are not at all meant to comprise a complete preschool-to-algebra curriculum; the Task Group merely aims to recognize the Critical Foundations for the study of Algebra, whether as part of a dedicated algebra course in the seventh, eighth, or ninth grade, or within an integrated mathematics sequence in the middle and high school grades. However, these Critical Foundations *do* deserve ample time in any mathematics curriculum. The foundations are presented in three distinct clusters of concepts and skills, each of which should incorporate the three aspects of learning noted here.

Fluency With Whole Numbers

By the end of the elementary grades, children should have a robust sense of number. This sense of number must include understanding place value, and the ability to compose and decompose whole numbers. It must clearly include a grasp of the meaning of the basic operations of addition, subtraction, multiplication, and division, including use of the commutative, associative, and distributive properties; the ability to perform these operations efficiently; and the knowledge of how to apply the operations to problem solving. Computational facility rests on the automatic recall of addition and related subtraction facts, and of multiplication and related division facts. It requires fluency with the standard algorithms for addition, subtraction, multiplication, and division. Fluent use of the algorithms not only depends on the automatic recall of number facts but also reinforces it. A strong sense of number also includes the ability to estimate the results of computations and thereby to estimate orders of magnitude, e.g., how many people fit into a stadium, or how many gallons of water are needed to fill a pool.

Fluency With Fractions

Before they begin algebra course work, middle school students should have a thorough understanding of positive as well as negative fractions. They should be able to locate both positive and negative fractions on the number line; represent and compare fractions, decimals, and related percents; and estimate their size. They need to know that sums, differences, products, and quotients (with nonzero denominators) of fractions are fractions, and they need to be able to carry out these operations confidently and efficiently. They should understand why and how (finite) decimal numbers are fractions and know the meaning of percentages. They should encounter fractions in problems in the many contexts in which they arise naturally, for example, to describe rates, proportionality, and probability. Beyond computational facility with specific numbers, the subject of fractions, when properly taught, introduces students to the use of symbolic notation and the concept of generality, both being integral parts of Algebra.

Particular Aspects of Geometry and Measurement

Middle-grade experience with similar triangles is most directly relevant for the study of Algebra: Sound treatments of the slope of a straight line and of linear functions depend logically on the properties of similar triangles. Furthermore, students should be able to analyze the properties of two- and three-dimensional shapes using formulas to determine perimeter, area, volume, and surface area. They should also be able to find unknown lengths, angles, and areas.

Benchmarks for the Critical Foundations

In view of the sequential nature of mathematics, the Critical Foundations of Algebra described in the previous section require judicious placement in the grades leading up to Algebra. For this purpose, the Task Group suggests the following benchmarks as guideposts for state frameworks for school districts. There is no empirical research on the placement of these benchmarks, but they find justification in a comparison of national and international curricula. The benchmarks should be interpreted flexibly, to allow for the needs of students and teachers.

Fluency With Whole Numbers

1) By the end of Grade 3, students should be proficient with the addition and subtraction of whole numbers.
2) By the end of Grade 5, students should be proficient with multiplication and division of whole numbers.

Fluency With Fractions

1) By the end of Grade 4, students should be able to identify and represent fractions and decimals, and compare them on a number line or with other common representations of fractions and decimals.
2) By the end of Grade 5, students should be proficient with comparing fractions and decimals and common percents, and with the addition and subtraction of fractions and decimals.
3) By the end of Grade 6, students should be proficient with multiplication and division of fractions and decimals.
4) By the end of Grade 6, students should be proficient with all operations involving positive and negative integers.
5) By the end of Grade 7, students should be proficient with all operations involving positive and negative fractions.
6) By the end of Grade 7, students should be able to solve problems involving percent, ratio, and rate, and extend this work to proportionality.

Particular Aspects of Geometry and Measurement

1) By the end of Grade 5, students should be able to solve problems involving perimeter and area of triangles, and all quadrilaterals having at least one pair of parallel sides (i.e., trapezoids).
2) By the end of Grade 6, students should be able to analyze the properties of two-dimensional shapes and solve problems involving perimeter and area. They should also be able analyze the properties of three-dimensional shapes and solve problems involving surface area and volume.
3) By the end of Grade 7, students should understand relationships involving similar triangles.

To address the question of whether the sequence of topics prior to formal algebra course work or how formal algebra course work affects achievement in algebra, the Task Group examined three related sub-questions. It first looked for evidence on the effectiveness of currently used elementary and middle school mathematics curricula (including their sequence of topics) for achievement in Algebra. It found no research demonstrating that a specific multigrade sequence of mathematics topics assures success in Algebra.

The Task Group also sought evidence on whether an integrated approach or a single-subject sequence might be more effective for formal algebra course work and more advanced mathematics course work. It found no clear body of research from which one may draw conclusions.

Finally, the Task Group sought to locate evidence on the benefits or disadvantages of teaching the content of an Algebra I course to a broad range of students before Grade 9. The Task Group found a positive relationship between taking Algebra in Grade 7 or 8 and later high school mathematics achievement, regardless of students' prior achievement and school and student characteristics.

Recommendations

This Task Group affirms that Algebra is the gateway to more advanced mathematics and to most postsecondary education. All schools and teachers of mathematics must concentrate on providing a solid mathematics education to all elementary and middle school students so that all of them can enroll and succeed in Algebra. Students need to be soundly prepared for Algebra and then well taught in Algebra, regardless of the grade level at which they study it. To improve the teaching of Algebra, the Task Group proposes the following eight recommendations:

1) The Task Group recommends that school algebra be consistently understood in terms of the Major Topics of School Algebra given in this report.

2) The Major Topics of School Algebra in this report, accompanied by a thorough elucidation of the mathematical connections among these topics, should be the main focus of Algebra I and Algebra II standards in state curriculum frameworks, in Algebra I and Algebra II courses, in textbooks for these two levels of Algebra whether for integrated curricula or otherwise, and in end-of-course assessments of these two levels of Algebra. The Task Group also recommends use of the Major Topics of School Algebra in revisions of mathematics standards at the high school level in state curriculum frameworks, in high school textbooks organized by an integrated approach, and in grade-level state assessments using an integrated approach at the high school, by Grade 11 at the latest.

3) Proficiency with whole numbers, fractions, and particular aspects of geometry and measurement are the Critical Foundation of Algebra. Emphasis on these essential concepts and skills must be provided at the elementary- and middle-grade levels. The coherence and sequential nature of mathematics dictate the foundational skills that are necessary for the learning of algebra. By the nature of algebra, the most important foundational skill is proficiency with fractions (including decimals, percent, and negative fractions). The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in Algebra can be expected.

4) The Benchmarks proposed by the Task Group should be used to guide classroom curricula, mathematics instruction, and state assessments. They should be interpreted flexibly, to allow for the needs of students and teachers.

5) International studies show that high-achieving nations teach for mastery in a few topics, in comparison with the U.S. mile-wide-inch-deep curriculum. A coherent progression, with an emphasis on mastery of key topics, should become the norm in elementary and middle school mathematics curricula. There should be a de-emphasis on a spiral approach in mathematics that continually revisits topics year after year without closure.

6) Federal and state policies should give incentives to schools to offer an authentic Algebra I course in Grade 8, and to prepare a higher percentage of students to study the content of such a course by the beginning of Grade 8. The word "authentic" is used here as a descriptor of a course that addresses algebra in the manner of Recommendation 2. Students must be prepared with the mathematical prerequisites for this course in the sense of Recommendation 3.

7) Publishers must ensure the mathematical accuracy of their materials. Those involved with developing mathematics textbooks and related instructional materials need to engage mathematicians, as well as mathematics educators, in writing, editing, and reviewing these materials.

8) Adequate preparation of students for Algebra requires their teachers to have a strong mathematics background. To this end, the Major Topics of School Algebra and the Critical Foundations of Algebra must be fundamental in the mathematics preparation of elementary and middle school teachers. Teacher education programs and licensure tests for early childhood teachers (preschool–Grade 3) should focus on the Critical Foundations of Algebra; for elementary teachers (Grades 1–5), on the Critical Foundations of Algebra and those algebra topics typically covered in an introductory Algebra course; and for middle school teachers (Grades 5–8), on the Critical Foundations of Algebra and all of the Major Topics of School Algebra.

I. Introduction

The learning of mathematics at the elementary- and middle-grade levels forms the basis for achievement in high school, college mathematics, and college courses using mathematics, and for the broad range of mathematical skills used in the workplace. Yet, on average, American students do not do as well on international mathematics tests as their peers in many developed countries. And while scores on the National Assessment of Educational Progress (NAEP) mathematics tests at Grades 4 and 8 are higher than they have ever been, a large majority of students do not score at the “proficient” and “advanced” levels (U.S. Department of Education, 2007). Nor are American students uniformly taught mathematics by teachers with an adequate grasp of the mathematics they teach (e.g., Loveless, 2004). There is broad agreement that mathematics education in the schools needs to be strengthened.

One of the major charges to the National Mathematics Advisory Panel concerned preparation for course work in algebra. A strong grasp of algebra is essential for successful participation in the contemporary American workforce; it is also necessary for entry into higher education and for the pursuit of advanced mathematics in general (e.g., Business Roundtable, 2006). The Panel was asked to make recommendations on “the critical skills and skill progressions for students to acquire competence in algebra and readiness for higher level mathematics” (Executive Order No. 13398). To address this particular charge, the Panel established a Task Group on Conceptual Knowledge and Skills. This report is the Task Group’s response to this charge and describes the essential mathematical concepts and skills that students should acquire prior to and during the study of algebra.

To guide its inquiry and deliberations, the Task Group formulated three major questions:

1) What are the major topics of school algebra?[2]
2) What are the essential mathematical concepts and skills that lead to success in Algebra and that should be learned as preparation for Algebra?[3]
3) Does the sequence of topics prior to algebra course work or for algebra course work affect achievement in Algebra?

II. Methodology

The Panel was charged with determining how to use “the results of research relating to proven-effective and evidence-based mathematics instruction” and making recommendations “based on the best available scientific evidence.” The Panel contracted with Abt Associates Inc. to survey the research literature for studies that addressed each task group’s major questions and met standards of methodological quality.

[2] “School algebra” is a term chosen to encompass the full body of algebraic material that the Task Group expects to be covered through high school, regardless of its organization into courses and levels.
[3] Algebra will be capitalized when it is referred to as a course.

The Task Group's literature review yielded some peer-reviewed and published studies that met standards of methodological quality and were relevant to the work of this Task Group, especially with respect to its third question. However, because of the small number of such studies, the Task Group decided to include other types of reports that presented the best available evidence on the topic of the conceptual knowledge and skills needed for success in Algebra. Thus it supplemented the review of research literature with reports by national organizations and government agencies, and with analyses and comparisons of state curriculum frameworks and school textbooks developed for the Task Group by the Institute for Defense Analyses Science and Technology Policy Institute (henceforth STPI).[4] Where there may be differences among these reports, studies, or analyses, the differences are so noted. The Task Group's recommendations on matters of definition and mathematical content were also guided by professional judgment.

III. Student Achievement in Mathematics

During a time of decline in test scores on the Scholastic Achievement Test (SAT) in the 1960s, Congress legislated for the creation of NAEP. It has been the only nationally representative and continuing assessment of what American students, through high school, know and can do in all major subject areas: reading, mathematics, science, writing, U.S. history, civics, geography, and the arts. Using nationally representative samples, NAEP provides information on the academic achievement of nationwide and state populations and subpopulations, not on individual students or schools.

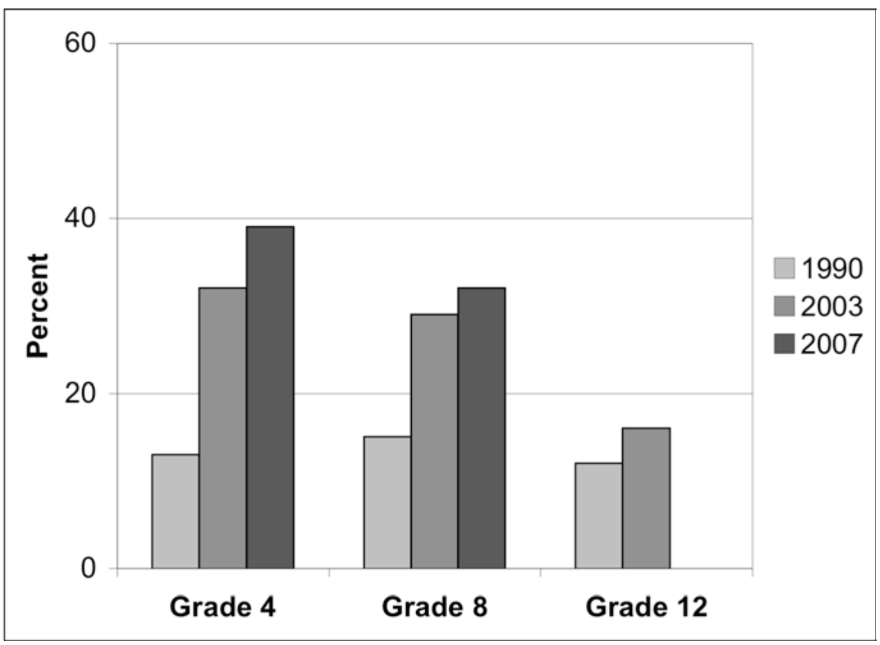
There are two types of NAEP tests. The long-term trend NAEP tests, which have been given since the late 1960s and early 1970s, assess students by age (9, 13, and 17). The main NAEP tests, which have been given since 1990, assess students in Grades 4, 8, and 12, instead of by age. For the long-term trend assessments, the same tests have been given under the same conditions since 1978. The main tests have been used since 1990 to create a second trend line that reflects more current practices in mathematics curriculum and assessment. The main NAEP tests in mathematics assess five areas: numbers and operations, measurement, geometry, data analysis and probability, and algebra; these tests have been governed by a framework that parallels the content strands in *Curriculum and Evaluation Standards for School Mathematics*, issued by the National Council of Teachers of Mathematics (NCTM) in 1989. The long-term trend NAEP tests in mathematics focus on essential concepts and skills in four areas: numbers and operations, measurement, geometry, and algebra. On both types of tests, students respond to questions of three types: multiple choice, short answer, and extended answer. However, on the long-term trend tests, most of the items are multiple choice. There is greater emphasis on extended responses on the main tests.

Student scores on both types of tests present a mixed picture of national achievement in mathematics, although the results in Grade 4 seem to allow for some optimism. As Figure 1 shows, average student scores on the main NAEP tests have increased considerably

[4] The Institute for Defense Analyses Science and Technology Policy Institute provided technical support to the National Mathematics Advisory Panel through a task order contract initiated in August 2006, Contract Number OIA-0408601 and Task Order OSTP-20-0001.

since 1990 at Grade 4 and to a lesser extent at Grade 8.[5] There have been increases at both grade levels on the long-term trend tests as well—not to the same extent as the increases on the main NAEP tests, but corroborating the trend. However, for Grade 4, U.S. results on the 1995 and 2003 Trends in International Math and Science Study (TIMSS) tests are exactly the same, raising questions about the large increase from 1990 to 2007 on the main NAEP tests (Mullis, Martin, Gonzalez, & Chrostowski, 2004).[6] Moreover, the percentage of students scoring at the proficient or advanced levels at Grade 4 and Grade 8 on the 2003 NAEP tests is well below 40%; at Grade 12, it is below 20%.[7] Not only is it not clear how valid the terms "proficient" and "advanced" are, it is also not clear how academically significant the gains are. Loveless (2004) conducted two analyses of the "arithmetic load" of 512 released items from NAEP's mathematics tests to determine their level of difficulty. Among items assessing "problem solving," he found that Grade 8 items were only slightly more difficult than the Grade 4 items, with many items testing arithmetic skills typically taught in Grades 1 and 2. He further noted that for these same items the algebra strand is the least-challenging content strand at both grades. In an analysis of calculator use on all released items in the number sense and algebra strands, he found that students are allowed to use calculators on items involving anything more difficult than whole numbers.

Figure 1: Percentage of Students At or Above Proficient in Mathematics Achievement On Main NAEP Test: 1990, 2003, and 2007



Source: Institute for Defense Analyses Science and Technology Policy Institute tabulations using the NAEP Data Explorer, available at: http://nces.ed.gov/nationsreportcard/nde/.

[5] See http://nces.ed.gov/nationsreportcard/nde/.
[6] The average was 518 in both years and the average age was 10.2 in both.
[7] Figure 1 shows the percent of students at or above proficient in mathematics achievement on the main NAEP test in 1990, 2003, and 2007 for Grade 4 and Grade 8. The percents in Grade 4 go from 13% in 1990, to 32% in 2003, to 39% in 2007. The percents in Grade 8 go from 15% in 1990, to 29% in 2003, to 32% in 2007. Grade 12 data are not available for 2007 as that grade was not tested that year.

It is also not clear that increased enrollment in advanced mathematics courses at the high school level is as academically significant as it appears. According to the U.S. Department of Education, the percentage of high school graduates completing Algebra II or higher rose from 39% in 1990 to 49% in 2005, with the number of students completing Calculus doubling from 1990 to 2005 (U.S. Department of Education, 1990–2005). Yet, test scores for high school students are flat on both the main and long-term trend NAEP tests, and there is no evidence that high school students are beginning their freshman year in college with stronger preparation for mathematics courses. In fact, almost one-fourth of the students enrolled in postsecondary education nationwide require placement in remedial mathematics courses, with the percentage varying by state and according to whether students are enrolled in 2-year or 4-year postsecondary institutions (U.S. Department of Education, 2004).

IV. What Are the Major Topics of School Algebra?

In response to this question, the Task Group reviewed the algebra topics addressed in several sources. They examined the algebra topics 1) in current state standards for Algebra I and Algebra II courses and for integrated curricula, 2) in current textbooks for school algebra and integrated mathematics, 3) in the algebra objectives in NAEP's 2005 Grade 12 mathematics assessment, 4) in the American Diploma Project's benchmarks for a high school exit test and its forthcoming Algebra II end-of-course test, and 5) in the algebra standards in Singapore's mathematics curriculum for Grades 7 through 10. The Task Group also developed its own list of major topics of school algebra. The Task Group presents its list first, together with a brief explanation of the logical connections among these topics, and then shows how this list compares with the algebra topics in these other sources.

A. The Major Topics of School Algebra

1. Introduction to the Topics

The Major Topics of School Algebra that were developed by the Task Group on Conceptual Knowledge and Skills are shown in this section. The teaching of Algebra, like the teaching of all school mathematics, must ensure that students are proficient in computational procedures, can reason logically and clearly, and can formulate and solve problems. For this reason, the topics listed below should not be regarded as a sequence of disjointed items, simply to be committed to memory. On the contrary, teachers and textbook writers should emphasize the connections as well as the logical progression among these topics. They comprise both core and foundational elements of school algebra—those elements needed for study of school algebra itself and those elements needed for study of more advanced mathematics courses. The total amount of time spent on covering them in single-subject courses is normally about 2 years, although algebra content may be and is often structured in other ways in the secondary grades. What is usually called Algebra I would, in most cases, cover the topics in Symbols and Expressions and Linear Equations and at least the first two topics in Quadratic Equations. The typical Algebra II course would cover the other topics, although the last topic in Functions (Fitting Simple Mathematical Models to Data), the last two topics in Algebra of Polynomials (Binomial Coefficients and the Binomial Theorem), and Combinatorics and Finite Probability are sometimes left out and then included in a precalculus course. It should be stressed that this list of topics reflects professional judgment as well as a review of what is in other sources.[8]

The Major Topics of School Algebra

Symbols and Expressions

- Polynomial expressions
- Rational expressions
- Arithmetic and finite geometric series

Linear Equations

- Real numbers as points on the number line
- Linear equations and their graphs
- Solving problems with linear equations
- Linear inequalities and their graphs
- Graphing and solving systems of simultaneous linear equations

Quadratic Equations

- Factors and factoring of quadratic polynomials with integer coefficients
- Completing the square in quadratic expressions
- Quadratic formula and factoring of general quadratic polynomials
- Using the quadratic formula to solve equations

Functions

- Linear functions
- Quadratic functions—word problems involving quadratic functions
- Graphs of quadratic functions and completing the square
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- Simple nonlinear functions (e.g., square and cube root functions; absolute value; rational functions; step functions)
- Rational exponents, radical expressions, and exponential functions
- Logarithmic functions
- Trigonometric functions
- Fitting simple mathematical models to data

Algebra of Polynomials

- Roots and factorization of polynomials
- Complex numbers and operations
- Fundamental theorem of algebra
- Binomial coefficients (and Pascal's Triangle)
- Mathematical induction and the binomial theorem

Combinatorics and Finite Probability

- Combinations and permutations as applications of the binomial theorem and Pascal's Triangle

[8] An expanded version of this overview can be found in an article, *The Major Topics of School Algebra* at http://math.berkeley.edu/~wu/ and http://math.harvard.edu/~schmid/.

2. Overview of School Algebra

Although the general reader may also find the following overview useful, the Task Group briefly explains in this section the major concepts of school algebra primarily for mathematics teachers and textbook publishers. Some common pitfalls in the classroom are also pointed out. Because textbooks often omit the mathematical connections between basic concepts and skills, a good part of the discussion is devoted to these connections. The Task Group believes that it is impossible to attain a basic understanding of algebra without a grasp of such connections. One consequence of this focus is that the discussion of word problems will be somewhat abbreviated. In no way, however, should this emphasis be interpreted to mean that problem solving is considered to be less important than these connections. Indeed, the solution of multistep word problems should be part of students' routine.

a. Symbols and Expressions

Without any doubt, the foundational skill of algebra is fluency in the use of symbols. A letter or symbol, e.g., x, is used to represent a number in the same way that the pronoun "she" is used to stand for a female. Unless the context makes it absolutely clear, it is necessary in each situation to state what kind of number x represents, e.g., a positive integer or a rational number. The importance of clearly specifying what each letter stands for cannot be overemphasized. For example, the commutative law of multiplication for whole numbers can be stated as follows: for any two whole numbers m and n, $mn = nm$. As is customary, we omit the multiplication symbol *x* when letters are being multiplied. The truth of this statement is easily verified by noting that a rectangular array of m rows of n dots has the same number of dots as an array of n rows of m dots. However, if one now allows m and n to stand for any two *real numbers*, then the truth of the same equality would be far less simple. It would depend on a precise definition of what a real number is.[9] Thus the meaning of a symbolic statement such as $mn = nm$ depends strongly on what the symbols m and n stand for. A symbolic statement in which the meaning of the symbols is not explained is not acceptable in mathematics. This is perhaps the most basic protocol in the use of symbols: **Always specify precisely what each symbol stands for.**

[9] In school mathematics, defining a real number as a point on the **number line** is a workable compromise. This is, in fact, one argument for emphasizing the importance of the number line in the school mathematics curriculum. For example, it is far from clear why $\sqrt{2}\sqrt{3} = \sqrt{3}\sqrt{2}$, because it is difficult to claim that one knows what these numbers $\sqrt{2}$ and $\sqrt{3}$ are. To say that $\sqrt{2}$ is the number whose square is 2 is to beg the question of how one can be sure there is such a number. If one tries to write down $\sqrt{2}$ by giving its decimal expansion, then one can give no more than its initial segment, e.g., 1.414213562373, but not the *complete* expansion. The same remark applies to $\sqrt{3}$. Such being the case, what does it mean to "multiply" these two numbers when one cannot even be certain of their very existence? And why are $\sqrt{2}\sqrt{3}$ and $\sqrt{3}\sqrt{2}$ equal?

A slightly different example in the use of symbols is the following. Suppose a student tries to solve the simple linear equation $2x = 3$ in x.[10] The solution, as is well known, is $\frac{3}{2}$. If the equation is $2x = 5$, then the solution is $\frac{5}{2}$. If the equation is $7x = 3$, then the solution is $\frac{3}{7}$. And so on. Students soon notice that, regardless of the specific numbers 2 and 3 used in the first equation, the numbers 2 and 5 in the second equation, or the numbers 7 and 3 used in the third equation, the solution is always the quotient of the second number by the first. This suggests the following symbolic presentation for presenting this idea succinctly and correctly: Given fixed numbers a and b, with $a \neq 0$, the solution to the equation $ax = b$ is $\frac{b}{a}$. Note that the number a in the equation $ax = b$ is the same number in the solution $\frac{b}{a}$. The same is true for the number b. When a symbol stands for exactly the same number throughout a discussion, it is called a **constant**. If a symbol is allowed to stand for a collection of numbers in a given discussion, such as the whole numbers or the rational numbers, that symbol is called a **variable**. For example, the m and n in the statement $mn = nm$, of the commutative law of multiplication for whole numbers, are variables.

When teaching introductory Algebra, it is important to give students the correct concept of a variable as a symbol used in the way described above; the common but misleading concept of a variable as "a quantity that varies" should be avoided. Unfortunately, the incorrect definition of a variable as "a quantity that varies" is usually presented right at the beginning of school algebra.

The following are some of the standard concepts associated with the use of symbols. Let x be any number. The number obtained by performing repeated arithmetic operations and taking roots, e.g., $\frac{1+x}{7}(\sqrt{5-x^2})$, is usually called a **symbolic expression** in x. Two of the most common kinds of symbolic expression are polynomials and rational expressions. An expression in x of the form $a_5x^5 + a_4x^4 + \ldots + a_1x + a_0$ where the $a_5, a_4, \ldots, a_0$ are constants, is called a **polynomial** of degree 5 in x with **coefficients** $a_5, \ldots, a_0$. A **rational expression** in x is a quotient of two polynomials in x, e.g., $\frac{x^6 - 7x^3}{x^4 + x + 2}$. If an equality between two symbolic expressions in x is valid for *all* possible values of x under discussion, then the equality is called an **identity**. For example, the first of the following is an identity for all positive integers n and the second is an identity for all numbers x, y:

$$1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1) \text{ for all positive integers } n,$$

$$(x + y)(x - y) = x^2 - y^2 \text{ for all numbers } x \text{ and } y.$$

[10] The concept of "solving an equation," which is explained later, is used only to illustrate one aspect of the use of symbols.

To verify these and other identities, one must remember that they are, above all, statements about *numbers* and, as such, all the knowledge about numbers that students bring to Algebra can be put to use for their verification. There is one caveat, however. Because students generally no longer have specific values of numbers for explicit arithmetic calculations (e.g., $15 \times 4 = 60$), all the calculations must now be carried out using only what is true for *all* numbers, regardless of what they are, i.e., the associative laws and commutative laws of addition and multiplication, and the distributive law.

One of the most important identities in introductory Algebra is

$$(x-y)(x^n + x^{n-1}y + \ldots + xy^{n-1} + y^n) = x^{n+1} - y^{n+1}$$

for all numbers x and y, and for all positive integers n. (This is a generalization of the second identity above.) The verification of this identity is a straightforward calculation. When students let $x = 1$, they get

$$(1-y)(1+y+y^2+\cdots+y^{n-1}+y^n) = 1-y^{n+1}$$

so that

$$1+y+y^2+\cdots+y^{n-1}+y^n = \frac{y^{n+1}-1}{y-1},$$

which is valid for all numbers y not equal to 1 and for all positive integers n. This is the **summation formula for finite geometric series**. This summation formula is important in mathematics and in both the natural and social sciences. The fact that it is an elementary result that can be taught at the beginning of school algebra is not generally recognized. It should be taught early because, when it is relegated to the end of Algebra II, as is done in the standard curriculum, it does not receive enough attention and is often omitted. The result is that many students go to college missing a critical piece of information.

b. Linear and Quadratic Equations

The most immediate application of the use of symbols is the solution of equations, usually linear and quadratic ones at the beginning, but more complicated ones as more sophisticated functions are defined. This discussion begins with **equations in one variable**, i.e., considering first those equations involving only one variable. The usual terminology of "solving" $3x - 1 = 4 + x$, for example, is an abbreviation of the longer statement of "determining for which numbers x the equality $3x - 1 = 4 + x$ is valid." Any such values of x are then called the **solutions** of the equation $3x - 1 = 4 + x$. For example, 5 is not a solution of $3x - 1 = 4 + x$, as $14 \neq 9$, but $\frac{5}{2}$ is a solution. Students will soon learn that the latter is the only solution possible.

Note that the equation $3x - 1 = 4 + x$ arises naturally if students try to answer the question: "What is the number if 1 less than 3 times the number is equal to 4 more than itself?" If students let x be this unknown number, then a direct symbolic transcription of the verbal data leads directly to $3x - 1 = 4 + x$. For this reason, the symbol x in an equation is sometimes called an **unknown**. In school algebra, it is therefore important to learn not only how to solve equations, but also to correctly transcribe verbal information into symbolic information. If standard textbooks on algebra serve as any indication, the latter objective has perhaps not been given the attention it deserves in the algebra classroom.

It is common for students to be taught to solve an equation by manipulating the symbolic expressions, where a symbolic expression is regarded as something distinct from anything they have ever encountered. A consequence is that ad hoc concepts, such as balancing an equation in x, have to be introduced to justify the method of solution. This line of thinking is not correct from a mathematical standpoint. The correct way to solve the equation $3x - 1 = 4 + x$, for example, involves nothing more than the consideration of numbers. The principle underlying the following solution method is applicable, not just to a linear equation, but to all equations. As a first step, students should begin by **assuming that there is a solution** to the equation. If the equation turns out to have no solution, then one would arrive at a contradiction and would know it had no solution. But if it does have a solution, one would be able to see what this solution must be. Therefore, one can call this putative solution x'. Then $3x' - 1 = 4 + x'$; note that this is an equality of two *numbers* and is therefore something with which students are completely familiar. What they do next is to try to deduce, under the assumption that they already have such a solution x', what this number x' has to be. Adding $-x' + 1$ to both sides of $3x' - 1 = 4 + x'$ to bring both terms involving x' to the left and the constants to the right, we obtain $(3x' - 1) + (-x' + 1) = (4 + x') + (-x' + 1)$, so that $2x' = 5$ and therefore $x' = \frac{5}{2}$.

Students have proved, *by using facts about numbers and nothing else*, that *if* there is a solution x', then it must be $\frac{5}{2}$. Notice that this does not say, as yet, that $\frac{5}{2}$ *is* a solution of $3x - 1 = 4 + x$. However, by verifying directly that $3(\frac{5}{2}) - 1 = 4 + \frac{5}{2}$, students reach the desired conclusion that, indeed, $\frac{5}{2}$ is a solution. It follows that $\frac{5}{2}$ is the only solution because students have already shown that any solution has to be $\frac{5}{2}$. The same reasoning shows why the solution of a **general linear equation,** $ax + b = cx + d$ (a, b, c, d being constants, and $a \neq c$), is the number $\frac{d - b}{a - c}$.

This method of solution is applicable to any equation, and, in particular, to a **quadratic equation** of one variable $ax^2 + bx + c = 0$ (a, b, c are constants and $a \neq 0$). Thus, assuming there is a solution (which is often called a **root**), students deduce what it must be, and then verify directly that any such possibility (depending on the values of the constants a, b, c, there may be one or two solutions) is indeed a solution. One particular detail of the solution is, however, of great interest, and it is the use of the technique of **completing the square.** This reduces the quadratic polynomial $ax^2 + bx + c$ to a simple form, from which the equation $ax^2 + bx + c = 0$ can be readily solved. The well-known **quadratic formula** is the result.

The skill of completing the square can be traced back to the Babylonians of 4,000 years ago. It is a useful skill in its own right and is the key idea that will lead to a complete clarification of the graph of a quadratic function, to be taken up in the section in this report on functions.

In the event that the quadratic polynomial $ax^2 + bx + c$ can be factored as a product, $ax^2 + bx + c = a(x - r_1)(x - r_2)$ for some numbers r_1 and r_2, then it is clear that r_1 and r_2 are the solutions of $ax^2 + bx + c = 0$. Less well known and less obvious is the *converse* of this statement, namely the fact that if r_1, r_2 are the solutions of $ax^2 + bx + c = 0$, then $ax^2 + bx + c = a(x - r_1)(x - r_2)$. This fact can be proven by using the quadratic formula, although it is not always done in textbooks. Of greater importance is the light this fact throws on the issue of factoring quadratic polynomials. When the coefficients *a*, *b*, *c* of $ax^2bx + c$ are integers, factoring such polynomials is sometimes elevated to an important skill in introductory Algebra. While some skill along this line is desirable, what this discussion has shown is that it is not necessary to emphasize it, because all such factoring can be done easily by using the quadratic formula to first locate the roots.

Next for consideration is the case of a **linear equation of two variables**, $ax + by = c$ where *a*, *b*, *c* are constants, and both *a* and *b* are not 0. A **solution** of the equation is by definition an ordered pair of numbers (x_0, y_0), so that they **satisfy the equation** in the sense that $ax_0 + by_0 = c$. This definition of a solution suggests that the collection of all solutions of $ax + by = c$ should be identified with a subset of the coordinate plane. Indeed, the **graph** of $ax + by + c = 0$ is defined to be the subset of the plane consisting of all solutions of the equation. One then proves that the graph is a straight line (or more simply, a **line**), and conversely, every straight line is the graph of an (essentially unique) linear equation of two variables. In the latter case, the equation is referred to as **the equation of the line**. The key ingredient in the proof of both facts is the concept of the **slope** of a line: Given a line *L* in the coordinate plane, its slope is the quotient $\frac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are *any* two points on the straight line *L*. The fact that this quotient remains the same no matter which two points, (x_1, y_1) or (x_2, y_2), are chosen is stated with not a hint of justification in almost all algebra textbooks currently in use. When students are denied access to this reasoning, it is difficult, if not impossible, for them to understand the relationship between a linear equation of two variables and its graph. The result is that many students consider the different forms of the equation of a line (e.g., point-slope form, two-point form) a mystery and are confused by related computations.

The proof that the definition of a slope of a line is independent of the choice of the two points depends on considerations of similar triangles. It is therefore vitally important that students be given the opportunity to become familiar with the basic facts of similar triangles before studying algebra. This should include the fact that corresponding sides of similar triangles are in proportion, or that if two triangles have two pairs of equal angles, they are similar. Students can defer the proofs of these theorems to a later course on Euclidean geometry, but they need to be comfortable using them. Students will commonly be asked to use certain theorems before they learn why the theorems are true, (e.g., lessons on the Pythagorean Theorem or the sum of angles of a triangle as 180 degrees). Mathematics learning does not have to be *formally* linear.

With the correct definition of slope available, students are in a position to understand the relationship between slopes of lines, and the concepts of parallelism and perpendicularity. This understanding has a strong bearing on the study of simultaneous linear equations in two variables. Using the precise definition of the graph of a linear equation of two variables, one can *prove* that the solution of a pair of simultaneous linear equations is the point of intersection of the graphs of the two equations. If the graphs are parallel, they do not intersect, and therefore there is no solution to the simultaneous equations. When the parallelism of the graphs is translated into the language of slope, students arrive at the criterion for the solvability of simultaneous equations in terms of the determinant of the system of equations.

Associated with a linear equation in two variables, $ax + by = c$, are the **linear inequalities** $ax + by \geq c$ and $ax + by \leq c$. It suffices to discuss one of them, say $ax + by \leq c$. Again, the need for the definition of the **graph of a linear inequality** should be emphasized: the graph of $ax + by \leq c$ is the collection of all points (x_0, y_0) in the coordinate plane so that $ax_0 + by_0 \leq c$. Excluding the case that $b = 0$ (which can be handled easily), the graph of $ax + by = c$ is then a non-vertical line, and one can then *prove* that the graph of $ax + by \leq c$ is all the points in the plane on or below the line $ax + by = c$ if $b > 0$ and on or above the line $ax + by = c$ if $b < 0$. Either of the latter is called a **half plane**. The fact that the graphs of linear inequalities are half planes of straight lines is needed for the solution of problems related to linear programming.

c. Functions

The concept of a function is a major building block of mathematics as a whole. Like most useful skills and concepts in mathematics, functions arise from the need for solving problems, more specifically, from the need for a tool to describe natural phenomena. For example, to arrive at a complete description of the temperature of a cup of freshly brewed coffee for the first 10 minutes after it has been poured, there is no alternative except to use a function f defined on the segment [0,10] from 0 to 10 on the number line, where the unit 1 represents 1 minute, so that for each t satisfying $0 \leq t \leq 10$, $f(t)$ gives the temperature of the coffee t minutes later. Many similar examples can be given so that students get to see the *need* for functions.

Given a function f of one variable, the **graph** of f is the collection of all the points in the plane of the form $(x, f(x))$ whenever $f(x)$ makes sense. Linear functions (of one variable) are those of the form $f(x) = cx + k$, where c and k are constants, and x is any number. Clearly the graph of f is the same as the graph of the linear equation in two variables $cx - y = -k$. It follows from the earlier discussion on page 10 that the graph of a linear function is a line. The linear functions of the form $f(x) = cx$, so-called **linear functions without constant term**, occupy a special place in school mathematics. It is only through the use of such functions that students can understand the usual discussion of so-called **proportional reasoning**. To see the relevance, notice that one can express $f(x) = cx$ in an equivalent way, as follows: for any two nonzero numbers x_1 and x_2, one has the proportional relationship $\frac{f(x_1)}{x_1} = \frac{f(x_2)}{x_2}$ (and the common value is c). Whenever proportional reasoning is called for, it means only one thing: A certain function turns out to be a linear function without constant term. A common flaw in the presentation of proportional reasoning is that certain functions that come up in such problems are *assumed automatically to be linear functions without constant term*. Unfortunately, such an assumption is sometimes not warranted. For example, the reasoning that "if five people need 12.5 liters of water for a camping trip, then eight people need 20 liters" would be accepted as a sensible rule of thumb in everyday life. In a classroom, however, it should be pointed out that the argument implicitly depends, in mathematical terms, on the assumption that every person needs exactly the same amount of water for the camping trip. Such an assumption, though unreasonable in other contexts, is necessary for the translation of the given data ("five people need 12.5 liters of water") into a mathematically solvable problem. The need for simplifying assumptions of this type must be explicitly pointed out to students.

After teaching linear functions, **quadratic functions** are next. These are functions f of the form $f(x) = ax^2 + bx + c$, where a, b, c are constants and x is any number. The **zeros** of f, i.e., the numbers x' so that $f(x') = 0$, are exactly the solutions of the quadratic equation $ax^2 + bx + c = 0$, which students will know are given by the quadratic formula. For more detailed information about f, one should consider the case of a positive a, as the case of a negative a is similar. By completing the square, one can rewrite f as $f(x) = a(x + p)^2 + q$, where p and q are constants, which can be explicitly determined in terms of a, b and c. In this form, one sees that (because a square is never negative) $f(x) \geq q$, and $f(x) = q$ exactly when $x = -p$. Thus the minimum value of f is q, and this happens exactly when $x = -p$. Another simple argument using $f(x) = a(x + p)^2 + q$ also shows that the graph of f is congruent to the graph of ax^2, and that the axis of (reflection) symmetry of the graph of f is the vertical line passing through $(-p, 0)$. Thus one can obtain all the essential information about f by simply applying the technique of completing the square.

The new information about quadratic functions greatly enlarges for students the scope of word problems. It is now relatively simple to find out among rectangles with a fixed perimeter, which has the biggest area. Word problems of the following type also become accessible: Two workmen, painting at a constant rate, can paint a house together in six days. In how many days can each paint it alone if it takes one of them three days longer than the other to get it done?

Beyond quadratic functions, there are not too many things one can say about **polynomial functions** [functions f so that $f(x)$ is a fixed polynomial in x] in general without first acquiring more advanced tools. For **rational functions** (quotients of polynomial functions), a new phenomenon is the emergence of the concept of an **asymptote** of its graph.

The next major classes of functions to be considered in school algebra are the exponential functions and the associated logarithmic functions. To solve $a > 0$ but $a \neq 1$, students have to make sense of the **exponential function** $h(x) = a^x$ for all numbers x. If x is a positive integer such as 7, the meaning of $h(7) = a^7$ is clear: $h(7) = aaaaaaa$. If x is a rational number, the meaning of a^x will have to be carefully defined, first for $x = 0$, then for x a fraction, and then for x a negative fraction. In school mathematics, all these considerations fall under the headings of rational exponents and laws of exponents. What needs to be pointed out is that the consideration of exponential functions is the main justification for teaching these topics. Once students know the meaning of $h(x)$ for all rational numbers x, in the context of school mathematics, this knowledge is already sufficient, and h is then known for all numbers x, rational or not. From the basic properties of rational exponents, it can be concluded that $h(x + x') = h(x)h(x')$ for all numbers x, x'. It is more common to express the last property directly as $a^{x+x'} = a^x a^{x'}$.

Suppose now $a > 1$. (One has to consider separately the case of an a so that $0 < a < 1$, but the reasoning is similar.) These same basic properties of rational exponents show that $h(x) = a^x$ is an increasing positive function. One can also present heuristic arguments as to why $h(x)$ goes to infinity as x goes to positive infinity, and $h(x)$ goes to 0 as x goes to negative infinity. In particular, $h(x)$ can be any positive number. Because the exponential function h is increasing and takes all positive values, it has an **inverse function** $\log_a s$, called the **logarithm with base a**, which is defined for all *positive* values s. Precisely, $\log_a s$ is the number x so that $a^x = s$ (i.e., $h(x) = s$). The fact that $h(x + x') = h(x)h(x')$ for all numbers x, x' now becomes $\log_a ss' = \log_a s + \log_a s'$ for all positive numbers s, s'. The graph of $\log_a$ is obtained from the graph of a^x by reflecting across the line $y = x$, but the proof of this fact requires some geometry.

Historically, the logarithmic functions were discovered before the exponential functions. Because $\log_a ss' = \log_a s + \log_a s'$, the multiplication ss' becomes (under $\log_a$) a sum, and with the compilation of so-called log tables, the computation of products of numbers becomes much more manageable. This was one reason that made the logarithm important before the advent of computers. Nowadays, of course, the logarithm is important for quite different reasons: $\log_a$ and the exponential function appear in nature in innumerable ways, and there is no way to avoid these two functions.

A final class of functions to be considered in school algebra is the set of **periodic functions** in general and the trigonometric functions—especially sine and cosine—in particular. A function f defined on the number line is periodic of period k for some positive number k if $f(x + k) = f(x)$ for all numbers x. The importance of such functions is clear once the periodic nature of many natural phenomena is realized.

d. Algebra of Polynomials

Thus far, a polynomial in a number x is a special kind of symbolic expression in x. One may therefore regard a polynomial as a function which assigns to each x the number given by that symbolic expression. At a more advanced level, a polynomial is a purely formal object and not a function. This is because algebra at an advanced level is an abstract study of structure and ceases being generalized arithmetic. School mathematics should introduce students to such abstract considerations at some point.

This introduction can include the following: Let X be a *symbol*, i.e., it no longer stands for a number. We will now prescribe how X should behave under addition and multiplication. Consider all sums of the form $f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots a_1 X + a_o$ where n is a whole number, the a_i's are constants (i = 0,1, ... , n), and $a_n \neq 0$. To avoid confusion with polynomials, such an $f(X)$ is called a **polynomial form**, but n will continue to be referred to as its **degree** and the a_n, ... a_0 as its **coefficients**. Now define the addition and multiplication of polynomial forms by requiring that the sum or product of any two polynomial forms should be equal to the polynomial form normally obtained if X were an ordinary number. Thus, for polynomial forms, addition and multiplication are associative, commutative, and distributive.

Introducing *polynomial forms* rather than just polynomials offers additional clarity by separating considerations of the coefficients a_i from those of the symbol X. As far as the arithmetic operations on polynomial forms are concerned, the preceding definitions imply that addition and multiplication will continue to be associative, commutative, and distributive if one replaces the real number coefficients by, for example, **complex number** coefficients. By doing that, the two kinds of polynomial forms are denoted by **R**[X] and **C**[X], where **R** and **C** denote, for obvious reasons, the real and complex numbers, respectively. On the other hand, X is now free to assume an existence of its own, and can now take on values other than real or complex numbers. In more advanced courses, for instance, X can be a square matrix, and polynomials of a square matrix are an integral component of linear algebra.

R and **C** share an important property, which is that in each case, every nonzero number has a **multiplicative inverse**, i.e., if $a \neq 0$, there is a b so that $ab = ba = 1$. Therefore the important **division algorithm**, which is the exact analog among polynomial forms of division with remainder among whole numbers, in either **R**[X] or **C**[X], is valid. Consequently, the **factor theorem** for real or complex polynomial forms is valid, which can be explained as follows. First, we say a number c (whether real or complex) is a **root** of a polynomial form $f(X)$ if the number $f(c)$ obtained by replacing X with c in $f(X)$ is equal to 0. Then the factor theorem states that given an $f(X)$ in **R**[X] (respectively, **C**[X]), a number c in **R** (respectively, **C**) is a **root** of $f(X)$ if and only if the linear polynomial form (X-c) in **R**[X] (respectively, **C**[X]) divides $f(X)$.

Having emphasized the formal similarity between **R**[X] and **C**[X], students can now be pointed to an essential difference between the two as a consequence of the difference between real and complex numbers. For complex numbers, the **fundamental theorem of algebra** applies: Every complex polynomial form of positive degree has a complex root. The proof of this theorem requires some advanced ideas, but as previously referenced in connection with similar triangles, the importance of the theorem justifies that school students learn it and use it even if they will not see how it is proved.[11] By repeated application of the factor theorem, students will see that *every complex polynomial form of degree n, where n is a positive integer, is equal to the product of n linear complex polynomial forms*. If $f(X)$ is a real polynomial form, i.e., $f(X)$ is an element in $\mathbf{R}[X]$, it can be regarded as a complex polynomial form and therefore is also the product of n linear complex polynomial forms [n is the degree of $f(X)$]. However, since $f(X)$ is a *real* polynomial form, one may wish to have a conclusion involving only real polynomial forms rather than complex ones. With this in mind, one applies the fundamental theorem of algebra with more care and concludes that *every polynomial form with real coefficients is the product of* real *linear polynomial forms and* real *quadratic polynomial forms without real roots.* The reasoning is very instructive.

So far, polynomial forms with one symbol X has been the focus, but there is no reason not to consider polynomial forms in more than one symbol. A case in point is the very natural question of whether there is a formula for $(X+Y)^n$, where X and Y are two symbols and n is a positive integer. The main impetus behind this question is the simple identity $(X+Y)^2 = X^2 + 2XY + Y^2$, which answers the question for $n = 2$. Additional effort by the students reveals that $(X+Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$. If one is persistent and computes the 4th, 5th, and even 6th powers, one would perceive a certain pattern and come up with a guess that the expansion of $(X+Y)^n$ must involve the so-called **binomial coefficients**. The precise result is the **binomial theorem**, and a common proof of this theorem uses the technique of **mathematical induction**. The latter is an integral part of school algebra.

e. Combinatorics and Finite Probability

In the process of proving the binomial theorem, one comes into contact with the basic properties of the binomial coefficients, the Pascal Triangle, and, consequently, simple facts about finite probability. Indeed, finite probability is fertile ground for applications of ideas in algebra as well as a rich source of problems.

B. Algebra Topics in Curriculum Sources

As mentioned in the introduction to Section V, the Task Group's judgment in formulating the Major Topics of School Algebra was informed by careful examinations of relevant source material. Now that the list itself has been presented and amplified by a discussion of important linkages among topics, the discussion returns to the various categories of source material. The following sections will show how the Task Group's Major Topics of School Algebra align with the actual content of current standards, teaching materials, and assessment tools.

[11] Students who take a college course in complex analysis will see then how it is proved.

1. State Standards for Algebra I and Algebra II

Because the responsibility for public education rests mainly with the states (and with local communities), requirements for the number of years of study in high school mathematics vary by state and across local communities within a state. Most states require 3 years or units of mathematics, a few require 2, and a small but growing number require 4 (Newton, Larnell, & Lappan, 2006). In addition, each state (except for Iowa, which has a draft of its secondary standards out for review) has its own mathematics standards or guidelines for Grades preschool through 8 or higher, as well as its own state tests. In the name of local control, local school districts also have their own curriculum standards or expectations, identifying what they believe to be the essential elements of mathematics content and instruction.

To determine common elements in algebra education across the states, the Task Group analyzed the content of state-based curriculum frameworks with specific attention to their standards, objectives, or course level expectations (CLEs). As of June 2006, the 22 states in Figure 2 provided standards for Algebra I and II courses. (A few other states provided only integrated curriculum standards at the high school level, while most states at that time did not provide any standards in high school mathematics.) However, in some of these states, (e.g., North Carolina) algebra standards are also offered as part of an integrated approach in the high school mathematics curriculum and may differ in coverage and level of difficulty from the algebra standards in their single-subject courses. An integrated approach may be generally defined as one in which the topics of high school mathematics are presented in some order other than the customary sequence in the Unites States of yearlong courses in Algebra I, Geometry, Algebra II, and Precalculus. In some states, algebra standards are offered only as part of an integrated approach rather than for single-subject courses.

Figure 2: States With Standards for Algebra I and II Courses

Alabama	Michigan
Arkansas	Mississippi
California	North Carolina
District of Columbia (counted as a state in NAEP)	Oklahoma
Florida	Oregon
Georgia	South Carolina
Hawaii	Tennessee
Indiana	Texas
Kentucky	Utah
Maryland	Virginia
Massachusetts	West Virginia

Source: The algebra curriculum of each state is available to the public on each state's department of education Web site. This figure was prepared for the Task Group by Institute for Defense Analyses Science and Technology Policy Institute in June 2006.

The frameworks for the Algebra I and Algebra II standards in these 22 states contain 300 different course level expectations (CLEs), which were organized into 31 major topics (Institute for Defense Analyses Science and Technology Policy Institute, in press, a). After tallying the frequency with which the CLEs occurred under each topic on a state-by-state basis, the Task Group found 13 broad topics included by at least 15 of these 22 states (see Table 1). Results for Algebra I and II were combined for this analysis.

Table 1: Frequency Counts for Broad Topics in 22 States' Standards for Algebra I and II Courses

Topic	Number of States (n = 22)
Linear equations and slope	21
Systems of equations	20
Evaluating, interpreting and representing data	19
Analyzing, interpreting and representing functions and relations	19
Inequalities	19
Real number operations	19
Solving quadratic problems	18
Exponents, roots, radicals, and absolute values	17
Operations with polynomials	16
Exponential functions and equations	16
Probability	16
Complex number operations	15
Rational equations and functions	15

Source: Institute for Defense Analyses Science and Technology Policy Institute, Brief Number 2, in press, a.

Note: Twenty-two states are shown, which represents the available information at the time.

Table 2 compares the Major Topics of School Algebra with the algebra topics in 20 Algebra I and Algebra II mathematics frameworks and in 3 high school integrated mathematics frameworks. These 23 sets of algebra topics come from 21 states that had content expectations of Algebra I or II, or an integrated math curriculum in Algebra I or II, or both. The Algebra I or II topics are in the 20 states from which the Task Group was able to obtain frameworks explicitly for Algebra. The algebra topics in the 3 sets of integrated mathematics frameworks come from 3 randomly selected states with integrated mathematics frameworks: Florida, North Carolina, and Georgia (Georgia no longer has standards for Algebra I and Algebra II courses). The comparisons do not reflect depth of treatment in the frameworks or the classroom. Nor do they necessarily reflect actual classroom content.

Table 2: Major Topics of School Algebra Covered by State Algebra or Integrated Mathematics Frameworks, by State and Two-Thirds Composite*

Major Topics of School Algebra	CA	GA (int)	AL	IN	MA	AR	FL	FL (int)	DC	MI	TN	MS	NC	NC (int)	VA	WV	UT	SC	HI	OK	OR	TX	MD	Two-thirds
Symbols and Expressions																								
Polynomial expressions																								
Rational expressions																								
Arithmetic and geometric sequences and series																								
Linear Relations																								
Real numbers as points on the number line																								
Linear equations and their graphs																								
Solving problems with linear equations																								
Linear inequalities and their graphs																								
Graphing and solving systems of simultaneous linear equations																								
Quadratic Relations																								
Factors and factoring of quadratic polynomials with integer coefficients																								
Completing the square in quadratic expressions																								
Quadratic formula and factoring of general quadratic polynomials																								
Using the quadratic formula to solve equations																								
Functions																								
Linear functions																								
Quadratic functions – word problems involving quadratic functions																								
Graphs of quadratic functions and completing the square																								
Polynomial functions (including graphs of basic functions)																								
Simple nonlinear functions (e.g., square and cube root functions; absolute value; rational functions; step functions)																								
Rational exponents, radical expressions, and exponential functions																								
Logarithmic functions																								
Trigonometric functions																								
Fitting simple mathematical models to data																								
Algebra of Polynomials																								
Roots and factorization of polynomial forms																								
Complex numbers and operations																								
Fundamental theorem of algebra																								
Binomial coefficients (and Pascal's Triangle)																								
Mathematical induction and the binomial theorem																								
Combinatorics and finite probability																								

Composite Key							
	Explicit		Not Found		Implicit/Incomplete		2/3 of States

*Integrated mathematics frameworks are identified as (int).

Source: Table created for the Task Group by Institute for Defense Analyses Science and Technology Policy Institute in June 2007.

Note: In this analysis, two-thirds of states included algebra as a single subject course or as integrated in the states' algebra framework.

2. Algebra I and Algebra II Textbooks

The Task Group examined an older algebra textbook as well as a number of current textbooks for Algebra I and II. Four of the five sets of Algebra I and II textbooks were national editions published by several major textbook publishers; a fifth set was the California edition published by Holt, Rinehart and Winston. The goal was to determine if there were any substantive differences between the content of a California edition and national editions. In addition, the Task Group also wanted to find out how an edition of the Algebra I and II textbooks authored by Mary Dolciani, dominant in the American market for many years, compared with current textbooks as well as with the Major Topics of School Algebra.

The Task Group's comparison of the major algebra topics in these five sets of Algebra I and Algebra II textbooks with the Major Topics of School Algebra appears in Appendix A. The Task Group's analysis of a textbook used in 1913 (see Figure 3) shows that the major topics, as derived from the table of contents, have remained much the same since the early 1900s except for the addition of material to modern textbooks addressing trigonometry, probability, and statistics (Institute for Defense Analyses Science and Technology Policy Institute, in press, a).

Figure 3: Topics in a 1913 High School Algebra Textbook

1. Definitions of Elementary Terms	14. Factors and Multiples
2. The Equation	15. Fractions
3. Addition	16. Ratio, Proportion, and Variation
4. Subtraction	17. Graphs of Linear Equations
5. Factoring	18. Systems of Linear Equations
6. Multiplication	19. Involution and Evolution
7. Division	20. Radicals and Exponents
8. Equations	21. Quadratic Equations
9. Type Products*	22. Systems of Quadratic Equations
10. Review and Extension of Processes	23. Graphs of Quadratic Equations
11. Exponents and Roots	24. Proportion, Variation and Limits
12. Logarithms	25. Series
13. Imaginary and Complex Numbers	26. Geometric Problems for Algebraic Solutions

*This chapter treats the identities $(a \pm b)^2 = a^2 \pm 2ab + b^2$, $(a+b)(a-b) = a^2 - b^2$ and related third-order identities, which are organized into "types."

Source: Young, 1913.

3. Singapore's Mathematics Curriculum for Grades 7–10

Singapore's fourth- and eighth-graders have consistently outperformed all other countries' students on the mathematics portion of the TIMSS (Gonzales et al., 2004). Singapore's compulsory secondary curriculum begins in Grade 7 and extends through the 10th year of schooling. Figure 4 shows the algebra standards addressed in Grades 7 through 10 (Singapore Ministry of Education, 2006). Table 3 shows the comparison of the list of Major Topics of School Algebra with the topics in Singapore's secondary curriculum.

Figure 4: Singapore's 2007 Algebra Standards for Grades 7–10

Algebraic Representation and Formulae
Using letters to represent numbers
Interpreting notations such as $\frac{3 \pm y}{5}$ *as* $(3 \pm y) \div 5$ *or* $\frac{1}{5 \times (3 \pm y)}$
Evaluation of algebraic expressions and formulae
Translation of simple "real-world" situations into algebraic expressions
Recognizing and representing number patterns (including finding an algebraic expression for the n^{th} term)
Algebraic Manipulation
Addition and subtraction of linear algebraic expressions
Simplification of linear algebraic expressions, e.g., $-2(3x - 5) + 4x$
Factorization of linear algebraic expressions of the form: $ax + ay$ (where a is a constant) and $ax + bx + kay + kby$ (where a, b and k are constants)
Expansion of the product of algebraic expressions
Changing the subject of a formula
Finding the value of an unknown quantity in a given formula
Recognizing and applying the special products
Factorization of algebraic expressions
Multiplication and division of simple algebraic fractions
Addition and subtraction of algebraic fractions with linear or quadratic denominator
Functions and Graphs
Cartesian coordinates in two dimensions
Graph of a set of ordered pairs
Linear relationships between two variables (linear functions)
Gradient of a linear graph as the ratio of the vertical change to the horizontal change (positive and negative gradients)
Graphs of linear equations in two unknowns
Graphs of quadratic functions and their properties: positive or negative coefficient of x^2, maximum and minimum points and symmetry
Sketching of the graphs of quadratic functions given in the form $y = \pm (x - p)^2 + q$ *and* $y = \pm (x - a)(x - b)$
Graphs of functions of the form y = ax^n where $n = -2, -1, 0, 1, 2, 3$, and simple sums of no more than three of these
Graphs of exponential functions $y = kax$ where a is a positive integer
Estimation of gradients of curves by drawing tangents
Solutions of Equations and Inequalities
Solving linear equations of one unknown (including fractional coefficients)
Solving simple inequalities (e.g., $3x \leq 5$)
Solving simple fractional equations that can be reduced to linear equations
Formulating a linear equation in one unknown to solve problems
Solving simultaneous linear equations in two unknowns by substitution and elimination methods, and graphical method
Solving quadratic equations in one unknown by factorization
Formulating a pair of linear equations in two unknowns or a quadratic equation in one unknown to solve problems
Solving quadratic equations in one unknown by: use of formula, completing the square for $y = x\ 2 + px + q$, or graphical methods
Solving fractional equations that can be reduced to quadratic equations
Solving linear inequalities in one unknown and representing the solution set on the number line

Continued on p. 3-21

Figure 4: Singapore's 2007 Algebra Standards for Grades 7–10, continued

Matrices
Display of information in the form of a matrix of any order
Interpreting the data in a given matrix
Product of a scalar quantity and a matrix
Problems involving the calculation of the sum and product (where appropriate) of two matrices
Ratio, Rate, and Proportion
Map scales (distance and area)
Direct and inverse proportion
Set Language and Notation
Use of set language
Union and intersection of two sets
Venn diagrams
Numbers and the Four Operations
Positive, negative, zero, and fractional indices
Laws of indices
Quadratic Equations and Inequalities
Conditions for a quadratic equation to have: two real roots, two equal roots, and no real roots
Related conditions for a given line to: intersect a given curve, be a tangent to a given curve, and not intersect a given curve
Solution of quadratic inequalities, and the representation of the solution set on the number line
Conditions for $r\ ax^2 + bx + c$ to be always positive (or always negative)
Relationships between the roots and coefficients of the quadratic equation $ax^2 + bx + c = 0$
Indices and Surds
Four operations on indices and surds
Rationalizing the denominator
Solving equations involving indices and surds
Polynomials
Multiplication and division of polynomials
Use of remainder and factor theorems
Factorization of polynomials
Solving cubic equations
Simultaneous Equations in Two Unknowns
Solving simultaneous equations with at least one linear equation, by substitution
Expressing a pair of linear equations in matrix form and solving the equations by inverse matrix method
Partial Fractions
Partial fractions including cases where the denominator is no more complicated than: $(ax + b)(cx + d)$, $(ax + b)(cx + d)^2$, and $(ax + b)(x^2 + c^2)$
Binomial Expansions
Use of the binomial theorem for positive integer n
Use of the notations $n!$ and $\binom{n}{r}$
Use of the general term $\binom{n}{r} a^{n-r}b^r$, $0 < r \leq n$
Exponential, Logarithmic, and Modulus Functions
Functions a^x, e^x, $\log_a x$, in x, and their graphs
Laws of logarithms
Equivalence of $y = a^x$ and $x = \log_a y$
Change of base of logarithms
Function x and graph of $f(x)$, where $f(x)$ is linear, quadratic, or trigonometric
Solving simple equations involving exponential, logarithmic, and modulus functions

Source: Singapore Ministry of Education, 2006.

Table 3: Comparison of Major Topics of School Algebra With Singapore's Secondary Mathematics Curriculum

Major Topics of School Algebra	**Singapore's Secondary Curriculum**
Symbols and Expressions	
Polynomial expressions	
Rational expressions	
Arithmetic and finite geometric series	
Linear Equations	
Real numbers as points on the number line	
Linear equations and their graphs	
Solving problems with linear equations	
Linear inequalities and their graphs	
Graphing and solving systems of simultaneous linear equations	
Quadratic Equations	
Factors and factoring of quadratic polynomials with integer coefficients	
Completing the square in quadratic expressions	
Quadratic formula and factoring of general quadratic polynomials	
Using the quadratic formula to solve equations	
Functions	
Linear functions	
Quadratic functions and word problems involving quadratic functions	
Graphs of quadratic functions and completing the square	
Polynomial functions, including graphs of basic functions	
Simple nonlinear functions (e.g., square and cube root functions, absolute value, rational functions, step functions)	
Rational exponents, radical expressions, and exponential functions	
Logarithmic functions	
Trigonometric functions	
Fitting simple mathematical models to data	
Algebra of Polynomials	
Roots and factorization of polynomials	
Complex numbers and operations	
Fundamental theorem of algebra	
Binomial coefficients (and Pascal's Triangle)	
Mathematical induction and the binomial theorem	
Combinatorics and Finite Probability	
Combinations and permutations as applications of the binomial theorem and Pascal's Triangle	

Note: A shaded cell shows agreement among the Major Topics of School Algebra and Singapore's secondary mathematics curriculum.

Source: The data in this table are STPI tabulations using data available from the Singapore Ministry of Education, 2007.

C. Algebra Topics in Assessment Sources

1. National Assessment of Educational Progress Test Objectives

NAEP developed two sets of mathematics objectives of interest to this Task Group, one for Grade 8 and one for Grade 12. The 2005 *NAEP Mathematics Framework for the 2005 National Assessment of Educational Progress* report describes a proposed special study at Grade 8 to “examine the breadth and depth of Grade 8 students’ understanding of proportionality and other fundamental topics in algebra” (National Assessment Governing Board, 2004). This document includes a list of objectives for Algebra and a list of objectives for proportionality that are to be used for this special assessment. However, because this assessment has not yet been scheduled, the Task Group did not examine its objectives. Figure 5 shows the algebra objectives addressed in NAEP’s Grade 12 assessment based on its 2005 assessment framework. Table 4 shows the comparison of the algebra topics in this set of objectives with the Major Topics of School Algebra.

Figure 5: Algebra Objectives for National Assessment of Educational Progress' Grade 12 Mathematics Assessment

Patterns, relations, and functions
Recognize, describe, or extend arithmetic, geometric progressions, or patterns using words or symbols
Express the function in general terms (either recursively or explicitly), given a table, verbal description, or some terms of a sequence
Identify or analyze distinguishing properties of linear, quadratic, inverse ($y = k/x$), or exponential functions from tables, graphs, or equations
Determine the domain and range of functions given various contexts
Recognize and analyze the general forms of linear, quadratic, inverse, or exponential functions (e.g., in $y = ax + b$, recognize the roles of a and b)
Express linear and exponential functions in recursive and explicit form given a table or verbal description
Algebraic representations
Translate between different representations of algebraic expressions using symbols, graphs, tables, diagrams, or written descriptions
Analyze or interpret relationships expressed in symbols, graphs, tables, diagrams, or written descriptions
Graph or interpret points that are represented by one or more ordered pairs of numbers on a rectangular coordinate system
Perform or interpret transformations on the graphs of linear and quadratic functions
Use algebraic properties to develop a valid mathematical argument
Use an algebraic model of a situation to make inferences or predictions
Given a "real-world" situation, determine if a linear, quadratic, inverse, or exponential function fits the situation (e.g., half-life bacterial growth)
Solve problems involving exponential growth and decay
Variables, expressions, and operations
Write algebraic expressions, equations, or inequalities to represent a situation
Perform basic operations, using appropriate tools, on algebraic expressions (including grouping and order of multiple operations involving basic operations, exponents, roots, simplifying, and expanding)
Write equivalent forms of algebraic expressions, equations, or inequalities to represent and explain mathematical relationships
Equations and inequalities
Solve linear, rational, or quadratic equations or inequalities
Analyze situations or solve problems using linear or quadratic equations, or inequalities symbolically or graphically
Recognize the relationship between the solution of a system of linear equations and its graph
Solve problems involving more advanced formulas [e.g., the volumes and surface areas of three-dimensional solids; or such formulas as: $A = P(1+r)^t$, $A = Pe^{rt}$]
Given a familiar formula, solve for one of the variables
Solve or interpret systems of equations or inequalities

Source: The data in this figure are Institute for Defense Analyses Science and Technology Policy Institute tabulations based on information from National Assessment Governing Board (2004).

Table 4: Comparison of the Major Topics of School Algebra With the 2005 NAEP Grade 12 Algebra Topics

Major Topics of School Algebra	NAEP Grade 12 Algebra Topics
Symbols and Expressions	
Polynomial expressions	
Rational expressions	
Arithmetic and finite geometric series	
Linear Equations	
Real numbers as points on the number line	
Linear equations and their graphs	
Solving problems with linear equations	
Linear inequalities and their graphs	
Graphing and solving systems of simultaneous linear equations	
Quadratic Equations	
Factors and factoring of quadratic polynomials with integer coefficients	(see note)
Completing the square in quadratic expressions	
Quadratic formula and factoring of general quadratic polynomials	
Using the quadratic formula to solve equations	
Functions	
Linear functions	
Quadratic functions and word problems involving quadratic functions	
Graphs of quadratic functions and completing the square	
Polynomial functions, including graphs of basic functions	
Simple nonlinear functions (e.g., square and cube root functions, absolute value, rational functions, step functions)	
Rational exponents, radical expressions, and exponential functions	
Logarithmic functions	
Trigonometric functions	
Fitting simple mathematical models to data	
Algebra of Polynomials	
Roots and factorization of polynomials	
Complex numbers and operations	
Fundamental theorem of algebra	
Binomial coefficients (and Pascal's Triangle)	
Mathematical induction and the binomial theorem	
Combinatorics and Finite Probability	
Combinations and permutations as applications of the binomial theorem and Pascal's Triangle	

Note: A shaded cell shows agreement among the Major Topics of School Algebra and 2005 NAEP.

"Factors and factoring of quadratic polynomials with integer coefficients" is subsumed under one of the Grade 12 objectives in the 2005 NAEP assessment framework. "Completing the square in quadratic expressions" is not explicitly a part of the 2005 framework at Grade 12. "Quadratic formula and factoring of general quadratic polynomials" appears partially in the 2005 Grade 12 framework. "Using the quadratic formula to solve equations" is not explicitly included as an objective at Grade 12. However, since there is an objective at Grade 12 on solving quadratic equations, students would have to utilize the quadratic formula or complete the square to solve a quadratic equation (P. Carr, personal communication, May 25, 2007).

Source: The data in this table are Institute for Defense Analyses Science and Technology Policy Institute tabulations using data available from National Assessment Governing Board, 2004.

2. American Diploma Project Benchmarks and Test Objectives

The American Diploma Project (ADP) Benchmarks describe the mathematics content and skills that ADP suggests all students should master by the time they leave high school if they are to be successful in college and work. Achieve Inc. developed these benchmarks based on research in colleges, universities, and high-performance workplaces across the country (Achieve Inc., 2007). The ADP benchmarks include five strands. The algebra strand is subdivided into six clusters, most of which include a number of benchmarks (up to eight). The six cluster headings are as follows:

1) Perform basic operations on algebraic expressions fluently and accurately.

2) Understand functions, their representations, and their properties.

3) Apply basic algebraic operations to solve equations and inequalities.

4) Graph a variety of equations and inequalities in two variables, demonstrate understanding of the relationships between the algebraic properties of an equation and the geometric properties of its graph, and interpret a graph.

5) Solve problems by converting the verbal information given into an appropriate mathematical model involving equations or systems of equations, apply appropriate mathematical techniques to analyze these mathematical models, and interpret the solution obtained in written form using appropriate units of measurement.

6) Understand the binomial theorem and its connections to combinatorics, Pascal's Triangle, and probability.

Currently, 30 states are working with Achieve Inc. to align their standards with the needs of college and work, as represented by the ADP Benchmarks. Thirteen of these states are also collaborating on an end-of-course Algebra II test that is based on the ADP Benchmarks and can serve as an indicator of readiness for credit-bearing college mathematics courses. Figure 6 shows the core algebra topics and the topics in the optional modules for this test.

Figure 6: Topics to Be Assessed in the American Diploma Project Algebra II End-of-Course Test

Topics in Core Test Modules	Topics in Optional Test Modules
Operations on Numbers and Expressions	**Data and Statistics**
Real numbers	Summarization and comparison of data sets
Complex numbers	Interpretation and communications through data
Algebraic expressions	**Probability**
Equations and Inequalities	Permutations, combinations, and probability
Linear equations and inequalities	Probability distributions
Nonlinear equations and inequalities	**Logarithmic Functions**
Polynomial and Rational Functions	Logarithmic expressions and equations
Quadratic functions	Logarithmic functions
Higher-order polynomial and rational functions	**Trigonometric Functions**
Exponential Functions	Trigonometric functions
Exponential functions	**Matrices**
Function Operations and Inverses	Matrix arithmetic
Function operations and composition	Solving systems of equations using matrices
Inverse functions	Matrix transformations
Piecewise functions	Vectors
	Conic Sections
	Conic sections
	Sequences and Series
	Arithmetic and geometric sequences and series
	Other types of iteration and recursion

Source: Achieve Inc., 2007.

Table 5 shows how ADP's high school benchmarks, the core topics in its Algebra II end-of-course test, and the topics in the optional modules for this Algebra II test compare with the Major Topics of School Algebra. There are three topic areas that require additional explanation. While the Major Topics of School Algebra lists these topics as algebra, the ADP addresses them in their Benchmarks as categories outside of algebra. Specifically, these areas include the following: 1) under the category of Linear Equations, real numbers as points on the number line are categorized by ADP as part of the Number Sense and Numerical Operations strand, and not in algebra as categorized by the Major Topics of School Algebra, 2) under the category of Functions, trigonometric functions are considered by ADP to be in the Geometry strand rather than algebra, and 3) under the category of Algebra of Polynomials, complex numbers are categorized as Number Sense and Numerical Operations (while operations are not in the Number Sense and Numerical Operations category) and not as algebra. Therefore, the lack of shading to represent agreement in these cases on what is considered algebra simply means that the ADP addresses them at other places in their Benchmarks.

Also of note, under the category of Algebra of Polynomials, the matching for those topics only refers to that fact the binomial theorem is discussed in ADP, but no proof by mathematical induction is required.

Table 5: Comparison of Major Topics of School Algebra With American Diploma Project's High School Algebra Benchmarks, Core Topics in Its Algebra II Test, and the Topics in the Optional Modules for Its Algebra II Test

Major Topics of School Algebra	ADP Algebra Benchmarks	ADP Algebra II Core	ADP Algebra II Optional
Symbols and Expressions			
Polynomial expressions			
Rational expressions			
Arithmetic and finite geometric series			
Linear Equations			
Real numbers as points on the number line			
Linear equations and their graphs			
Solving problems with linear equations			
Linear inequalities and their graphs			
Graphing and solving systems of simultaneous linear equations			
Quadratic Equations			
Factors and factoring of quadratic polynomials with integer coefficients			
Completing the square in quadratic expressions			
Quadratic formula and factoring of general quadratic polynomials			
Using the quadratic formula to solve equations			
Functions			
Linear functions			
Quadratic functions, word problems involving quadratic functions			
Graphs of quadratic functions and completing the square			
Polynomial functions, including graphs of basic functions			
Simple nonlinear functions (e.g., square and cube root functions, absolute value, rational functions, step functions)			
Rational exponents, radical expressions, and exponential functions			
Logarithmic functions			
Trigonometric functions			
Fitting simple mathematical models to data			
Algebra of Polynomials			
Roots and factorization of polynomials			
Complex numbers and operations			
Fundamental theorem of algebra			
Binomial coefficients and Pascal's Triangle			
Mathematical induction and the binomial theorem			
Combinatorics and Finite Probability			
Combinations and permutations as applications of the binomial theorem and Pascal's Triangle			

Note: A shaded cell shows agreement among the Major Topics of School Algebra and the three ADP categories.

Source: Table created for the Task Group by Institute for Defense Analyses Science and Technology Policy Institute from information available from Achieve Inc., 2007.

D. Comparisons

To show how the 27 Major Topics of School Algebra first listed on page 5 compare with current practices, they were matched against algebra topics listed in 1) U.S. state standards for Algebra I and Algebra II courses, 2) current algebra textbooks, 3) Singapore's 2007 algebra standards for Grades 7 through 10, 4) NAEP's assessment objectives for its 2005 Grade 12 test, and 5) the ADP benchmarks for a high school exit test, its core Algebra II end-of-course test and its optional modules for this test. In Tables 2, 3, 4, and 5, the Major Topics of School Algebra served as row headings and the comparison sources served as column headings. The corresponding cell was shaded or filled in when a comparison source clearly included that specific Major Topic of School Algebra. It is important to note that a shaded cell simply means coverage, not extent of coverage.

Potential sources of error in this analysis are the different ways in which the 27 topics may be worded in each document. Some topics do not appear to be covered in a comparison source, but they may be covered under another topic in the comparison source. For example, although none of the comparison sources explicitly covers polynomial functions, some sources include these functions under such headings as rational equations and functions or operations with polynomials. The level of detail possible for this analysis did not allow for reconciliation of misalignments of this type.

As Tables 2 and 3, and Appendix A show, the three comparison sources providing topics for algebra course work (state algebra standards, algebra textbooks, and Singapore's secondary mathematics curriculum) include most, if not all, of the Major Topics in School Algebra. Overall, almost all the Major Topics of School Algebra can be found in the state standards for Algebra I and II; moreover, a majority of the topics appear in at least two-thirds of the available frameworks examined. In Singapore's secondary mathematics curriculum, only two topics do not appear to be covered, and they are the fundamental theorem of algebra, and combinatorics and finite probability (Table 3), although it is possible that these topics are covered after Grade 10. All the Major Topics of School Algebra appear in almost every set of Algebra I and II textbooks that the Task Group examined (Appendix A), whether national or state editions. In addition, all the Major Topics of School Algebra were addressed in the Dolciani-authored Algebra I and II textbooks (Dolciani, Swanson, & Graham, 1986; Dolciani, Sorgenfrey, Brown, & Kane, 1988). A striking and significant difference lies in the number of topics and page length of all current Algebra I textbooks, each of which has close to 1,000 pages and attempts to address far more topics than the more focused and much slimmer texts of 20 years ago. For example, the now out-of-print Dolciani algebra textbooks, which were among the most widely used textbooks of their day, had far fewer pages and focused on far fewer topics. It is not clear how many of the topics in current Algebra I and II textbooks students can realistically study in the course of one year, and, more importantly, to what depth they study the major algebra topics.

On the other hand, comparisons with sources that provide assessment standards or objectives show gaps. The NAEP algebra objectives for its current Grade 12 test do not include many of the Major Topics of School Algebra, such as real numbers as points on a number line, all the topics listed under the algebra of polynomials, and combinatorics and finite probability (Table 4). However, the National Assessment Governing Board has revised the Grade 12 objectives for the mathematics test to be administered in 2009. Several of the Major Topics of School Algebra not included on the 2005 NAEP test will be assessed on the 2009 test, including arithmetic and finite geometric series, logarithmic functions, trigonometric functions, binomial coefficients (and Pascal's Triangle), and mathematical induction and the binomial theorem (P. Carr, personal communication, May 24, 2007). It is important to remember that these assessment objectives were designed, as are all NAEP assessment objectives, expressly for the purpose of assessment (in this case, of high school mathematics), not for the development of curriculum frameworks.

A comparison with the ADP's core topics for its Algebra II end-of-course test (Table 5) also shows gaps. The ADP's list of core topics for its Algebra II end-of-course test does not explicitly include such subjects as arithmetic and finite geometric series or linear equations and their graphs, which are typically taught prior to Algebra II. But the list also omits some traditional Algebra II topics, such as logarithmic functions, the binomial theorem and Pascal's Triangle, and mathematical induction. The gaps are fewer when the topics in the optional modules for this test are included. In particular, the optional modules do cover arithmetic logarithmic functions, fitting simple mathematical models to data, the binomial theorem and Pascal's Triangle, mathematical induction, and combinatorics and finite probability. However, these topics will be assessed only if a state chooses to test the module in which they appear.

In sum, most of the Major Topics in School Algebra are addressed in state algebra standards for Algebra I and Algebra II, albeit inconsistently across the 21 states. They are all addressed in almost all the algebra textbooks that were examined. And they are addressed almost completely in Singapore's algebra standards for Grades 7 through 10. The Major Topics of School Algebra have the least amount of coverage in assessment objectives, for NAEP's current Grade 12 test and for ADP's forthcoming Algebra II end-of-course test of core topics.

E. Observations Regarding Rigor in Algebra Textbooks

The Task Group commissioned a systematic examination of leading Algebra I and Algebra II textbooks for mathematical accuracy. The results of the survey are described in Appendix B. They reveal a systemic problem: Textbook publishers, their authors, and editorial staff do not pay sufficient attention to mathematical accuracy. It should be emphasized that the Task Group is not asking for rigor in a formal mathematical sense. The mathematics should be presented in an age-appropriate fashion, yet be clear and accurate. Circular definitions or the omission of a definition of an important notion being introduced must be avoided, and can be avoided without making the material less accessible.

Many of the problems uncovered by the textbook examination will not be apparent to most students, or even to their teachers. However, such problems tend to affect students' learning in both overt and subtle ways. Mathematical reasoning, accuracy, and clarity of thought are learned by example. Mathematically flawed textbooks hinder this learning process.

V. What Are the Essential Mathematical Concepts and Skills That Lead to Success in Algebra and Should Be Learned As Preparation for Algebra?

The mathematics that children learn from preschool through the middle grades provide the basic foundation for Algebra and more advanced mathematics course work. What is taught at particular grade levels is determined at the local and state level, and reflects the interests of a variety of national, state, and local agencies and organizations, as well as parents and the general public. In the past, there has been no research base to guide them. However, the results of TIMSS and other international tests showing student achievement across the participating countries have led to international comparisons of curricula and provided much information on what high-achieving countries teach their students in elementary and middle school.

To suggest what essential concepts and skills should be learned as preparation for algebra course work, the Task Group reviewed the skills and concepts listed 1) in the Grades 1 through 8 curricula of the highest-performing countries on TIMSS, 2) in NCTM's *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (National Council of Teachers Mathematics, 2006), 3) in Grades K through 8 in the six highest-rated state curriculum frameworks in mathematics, 4) in a 2007 American College Testing (ACT) survey (American College Testing, 2007), and 5) in a Panel-sponsored survey of 743 teachers of introductory Algebra across the country who were asked what students need to learn to be prepared for success in Algebra (Hoffer, Venkataraman, Hedburg, & Shagel, 2007). The Task Group also took into consideration the structure of mathematics itself, which requires teaching a sequence of major topics. Based on these sources and considerations, the Task Group proposes three clusters of concepts and skills, defined later as the Critical Foundations on page 40, reflecting their judgment about the most essential mathematics for students to learn thoroughly prior to algebra course work. It should be noted that there is no direct empirical evidence to support the effectiveness of any lists discussed in this section for success in algebra course work.

A. International Approaches to Pre-Algebra Education

1. Mathematics Topics Taught in Grades 1 Through 8 in the TIMSS Top-Performing Countries

One of the richest sources of information on mathematics curricula in other countries is the work of William Schmidt and his colleagues, who used data drawn from TIMSS. Schmidt, Houang, and Cogan (2002) compared the mathematics topics and the grade levels at which they were taught in the six highest-performing countries (Singapore, Japan, Korea, Hong Kong, Flemish Belgium, and the Czech Republic), which they called the "A+ countries." Figure 7 shows the composite mathematics curriculum profile for Grades 1 through 8 from a paper that is slightly revised from 2002 (Schmidt et al., 2002; Schmidt & Houang, 2007).

Figure 7: Mathematics Topics Intended From Grade 1 to Grade 8 by a Majority of TIMSS 1995 Top-Performing Countries

Mathematics Topics Intended	Grade 1	Grade 2	Grade 3	Grade 4	Grade 5	Grade 6	Grade 7	Grade 8
1. Whole Number: Meaning	●	●	●	●	●			
2. Whole Number: Operations	●	●	●	●	●			
3. Measurement Units	●	●	●	●	●	●	●	
4. Common Fractions			●	●	●	●		
5. Equations & Formulas			●	●	●	●	●	●
6. Data Representation and Analysis			●	●	●	●		●
7. 2-D Geometry: Basics			●	●	●	●	●	●
8. 2-D Geometry: Polygons and Circles				●	●	●	●	●
9. Measurement: Perimeter, Area and Volume				●	●	●	●	●
10. Rounding and Significant Figures				●	●			
11. Estimating Computations				●	●	●		
12. Whole Numbers: Properties of Operations				●	●			
13. Estimating Quantity and Size				●	●			
14. Decimal Fractions				●	●	●		
15. Relation of Common and Decimal Fractions				●	●	●		
16. Properties of Common and Decimal Fractions					●	●		
17. Percentages					●	●		
18. Proportionality Concepts					●	●	●	●
19. Proportionality Problems					●	●	●	●
20. 2-D Geometry: Coordinate Geometry					●	●	●	●
21. Geometry: Transformations						●	●	●
22. Negative Numbers, Integers, and Their Properties						●	●	
23. Number Theory							●	●
24. Exponents, Roots and Radicals							●	●
25. Exponents and Orders of Magnitude							●	●
26. Measurement: Estimation and Errors							●	
27. Constructions Using Straightedge and Compass							●	●
28. 3-D Geometry							●	●
29. Geometry: Congruence and Similarity								●
30. Rational Numbers and Their Properties								●
31. Patterns, Relations and Functions								●
32. Proportionality: Slope and Trigonometry								●

Note:

● Individual topics intended by more than half of the top-achieving countries: Singapore, Japan, Korea, Hong Kong, Flemish Belgium, and the Czech Republic.

Collection of topics intended by more than half of top-achieving countries.

Source: Schmidt & Houang, 2007.

2. Differences in Curriculum Approaches Between TIMSS Top-Performing Countries and the United States

In 2003, the International Association for the Evaluation of Educational Achievement reported on its survey of educators in Singapore, Japan, Flemish Belgium, Chinese Taipei, and Korea to learn more about their mathematics curriculum in Grades 4 and 8 (Mullis et al., 2004). The mathematics curricula in these countries show different entry and exit points for many topics. That is, they introduce and complete the study of many topics at different grade levels. For example, patterns of numbers or shapes is taught between Grades 1 and 5 in Singapore, followed by missing numbers in an equation between Grades 2 and 5. In contrast, these concepts are generally not studied until Grade 4 in Japan or Chinese Taipei, and even later in Flemish Belgium.

There seem to be two major differences between the curricula in top-performing countries and U.S. curricula: in the number of mathematical concepts or topics presented at each grade level and in the expectations for learning. U.S. curricula typically include many topics at each grade level, with each receiving relatively light development, while top-performing countries present fewer topics at each grade level but in greater depth. In addition, U.S. curricula generally review and extend at successive grade levels many (if not most) topics already presented at earlier grade levels, while the TIMSS top-performing countries are more prone to expect proficiency in what is taught at each grade level. These critical differences distinguish a spiral curriculum (common in many subjects in U.S. curricula) from one built on proficiency—a curriculum that expects proficiency in the topics that are presented before more complex or difficult topics are introduced.

In addition, the mathematics curricula in these top-performing countries show study of topics that are not in many U.S. curricula for these grade levels. For example, simple linear equations and simultaneous linear equations are taught in Grades 7 through 8 in the Singapore curriculum, Grades 7 through 9 in Japan, and Grade 8 in Chinese Taipei. A majority of U.S. state curriculum frameworks do not present these algebra concepts in Grades 7 and 8, although other algebra topics are taught in some states before Grade 9 (Newton et al., 2006).[12]

It is important to note that these A+ countries, as well as many other countries, teach Algebra in Grade 8, if not earlier. For example, Singapore begins study of Algebra in Grade 7. Schmidt et al. (2002) note that the “A+ composite curriculum portrays an evolution from an early emphasis on arithmetic in Grades 1 through 4 to more advanced Algebra and Geometry beginning in Grades 7 and 8. Grades 5 and 6 serve as a transitioned stage in which such topics as proportionality and coordinate geometry are taught, providing a bridge to the study of Algebra and Geometry” (p. 5).

[12] According to Newton et al.’s analysis of state frameworks in 2006, 12 states specify two-step equations as a grade-level expectation in Grade 7, and 16 do so in Grade 8. It is not clear what the overlap is among states.

B. National Approaches to Pre-Algebra Education

1. National Council of Teachers of Mathematics Curriculum Focal Points

In September 2006, NCTM released *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (*Focal Points*), a document that drew on an analysis of national and international programs to respond to a need for coherence in U.S. mathematics curricula and to offer direction for teachers. The *Focal Points* are suggested grade-level areas of emphasis—the concepts, skills, and procedures that connect important mathematics topics from grade to grade, and form the foundation for more advanced mathematics, beginning with Algebra. NCTM's Web site[13] provides a description of the purposes for the *Focal Points*, their grade-level connections, and the mathematics defining them.

Figure 8 shows a comparison of the *Focal Points* with the composite curriculum of the A+ countries (Schmidt & Houang, 2007). After comparing NCTM's 1989 standards with this composite curriculum, Schmidt and Houang (2007) judge the *Focal Points* as representing a "movement toward more coherent standards." However, as Figure 8 shows, the *Focal Points* recommends study in the primary grades of much more than arithmetic topics (e.g., 2-D and 3-D geometry; transformations; and patterns, relations, and functions). As noted earlier, and as the composite curriculum in Figure 7 shows, the A+ countries concentrate on arithmetic topics in Grades 1 and 2.

[13] http://www.nctm.org/standards/focalpoints.aspx?id=282

Figure 8: Mathematics Topics Intended From Grade 1 to Grade 8 in the 2006 NCTM *Focal Points* Compared With the Topics Intended by a Majority of TIMSS 1995 Top-Achieving Countries

Mathematics Topics Intended	Grade 1	2	3	4	5	6	7	8
1. Whole Numbers: Meaning	● ░	● ░	● ░	● ░	● ░	●		
2. Whole Numbers: Operations	● ░	● ░	● ░	● ░	● ░			
3. Measurement: Units	● ░	● ░	● ░	● ░	● ░	● ░	● ░	●
4. Common Fractions			● ░	● ░	● ░	● ░	●	
5. Equations & Formulas			░	● ░	● ░	● ░	● ░	● ░
6. Data Representation & Analysis	●		● ░	● ░	● ░	░	● ░	● ░
7. 2-D Geometry: Basics			● ░	● ░	░	░	░	● ░
8. 2-D Geometry: Polygons & Circles	●	●	●	● ░	● ░	░	● ░	● ░
9. Perimeter, Area & Volume			●	● ░	● ░	● ░	● ░	● ░
10. Rounding and Significant Figures				░	░			
11. Estimating Computations		●	●	● ░	● ░	░		
12. Properties of Operations	●	●	●	● ░	● ░			
13. Estimating Quantity and Size				● ░	░			
14. Decimal Fractions				● ░	● ░	● ░		
15. Relationships of Common & Decimal Fractions			●	● ░	░	● ░	●	
16. Properties of Common and Decimal Fractions					● ░	● ░		
17. Percentages					░	░	●	
18. Proportionality Concepts		●			░	● ░	● ░	● ░
19. Proportionality Problems					░	● ░	● ░	● ░
20. 2-D Coordinate Geometry					● ░	░	░	● ░
21. Transformations	●		●	●		░	░	░
22. Negative Numbers, Integers and Their Properties					●	░	● ░	
23. Number Theory	●				●		● ░	░
24. Exponents, Roots and Radicals							░	● ░
25. Exponents and Orders of Magnitude							░	● ░
26. Estimation and Errors		●	●		●		░	
27. Constructions w/ Straightedge and Compass							░	░
28. 3-D Geometry	●	●			●		● ░	● ░
29. Congruence and Similarity	●		●					● ░
30. Rational Numbers and Their Properties							●	░
31. Patterns, Relations and Functions	●	●	●	●	●	●		● ░
32. Slope and Trigonometry							●	● ░

Note:
░ Collection of topics intended by more than half of top-achieving countries
● 2006 NCTM Focal Points (*Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*)

Source: Schmidt & Houang, 2007.

2. Skills and Concepts in the Six Highest-Rated State Curriculum Frameworks

Also compared were the mathematics topics in the six highest-rated state curriculum frameworks, as judged by Klein et al. (2005), with the curriculum profile of the A+ countries (Schmidt & Houang, 2007). The Task Group used the highest-rated frameworks as noted in Klein et al. because this evaluation is the most comprehensive and in-depth review of state mathematics frameworks to date. The reviewers rated each state's standards for content, clarity, reason, and negative qualities, assigning different weights to each criterion for the overall assessment. The six highest-rated states are, in rank order, California, Indiana, Massachusetts, Alabama, New Mexico, and Georgia. A shaded cell in Table 6 indicates that the topic in the left-hand column appears in the mathematics standards of at least one of the six states at the given grade level. The 27 topics in Table 6 were among the 30 developed by the Task Group for examining state curriculum frameworks in mathematics and reflect the topics in the Schmidt et al. analysis (Institute for Defense Analyses Science and Technology Policy Institute, in press, b). They have been placed as closely as possible in the order in which the mathematics topics in the composite curriculum for the A+ countries first appear (Figure 7), an order that presumably reflects increasing difficulty or complexity.

Table 6: K Through 8 Grade-Level Expectations in the Six Highest-Rated State Curriculum Frameworks in Mathematics Compared With the Topics Intended by a Majority of TIMSS 1995 Top-Performing Countries*

		Grade								
Topics		**K**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	Whole Number Meaning	**6**	**5**	**5**	**4**	**6**	**2**			
2	Whole Number Operations	**3**	**4**	**5**	**6**	**3**				
3	Measuring and Units of Measurement	**6**	**6**	**2**	**3**	**4**	**2**	**6**		
4	Common Fractions	**3**	**1**	**4**	**4**	**5**	**3**			
5	Equations and Formulas				**2**	**2**	**2**	**4**	**6**	**6**
6	Collecting, Evaluating, Interpreting, and Representing Data	**4**	**5**	**5**	**3**	**4**	**5**	**5**	**3**	**4**
7	2-D Geometry Basics		**4**	**5**	**5**	**4**	**2**	**3**		
8	Polygons and Circles	**6**	**6**		**1**	**1**	**2**	**2**		**1**
9	Perimeter, Area, and Volume				**2**	**2**	**5**	**4**	**5**	**4**
10	Rounding				**4**	**5**	**4**			
11	Estimating Computations and Determine Reasonableness				**3**	**3**	**3**	**3**	**1**	
12	Properties of Whole Number Operations				**5**	**2**	**1**	**3**	**5**	**4**
13	Decimal Fractions and Decimals			**6**	**2**	**4**	**1**	**1**	**2**	**4**
14	Percentages						**5**	**4**	**2**	**1**
15	Ratios and Proportions							**2**	**1**	**2**
16	2-D Coordinate Geometry				**3**	**4**	**2**	**4**	**2**	
17	Geometric Transformation				**1**	**3**	**1**	**1**	**3**	**3**
18	Integers & Their Properties						**4**	**4**		
19	Number Theory						**5**	**3**	**1**	**1**
20	Exponents, Roots, Radicals, and Absolute Values							**3**	**5**	**6**
21	3-D Geometry				**6**	**2**	**2**	**1**	**3**	**1**
22	Congruence and Similarity		**1**	**2**	**2**	**5**	**2**	**2**	**4**	**3**
23	Rational Numbers and Their Properties							**2**	**6**	**5**
24	Patterns	**6**	**3**	**3**	**3**	**3**	**4**	**4**	**4**	**2**
25	Functions and Relations							**2**	**2**	**4**
26	Slope and Rates of Change							**1**	**2**	**3**
27	Probability		**1**	**3**	**3**	**3**	**3**	**5**	**1**	**4**

Note: A shaded cell indicates that at least one state specified an objective in this area at the given grade level.

STPI used 30 topics for its original analysis of these six state frameworks. Twenty-five of these topics agree with the TIMSS topics in Figure 7. Three of the remaining five topics were eliminated chiefly because they appeared to overlap with existing TIMSS topics. It is important to note that how STPI defined its 30 topics may differ from how these topics were defined by Schmidt et al. (2002) since it is not completely clear from the latter's writings how the 30 topics in Figure 7 were defined. A shaded cell indicates that at least one state specified an objective in this area at the given grade level.

The numbers in the shaded cells refer to the number of states within the six states that list that topic in their curriculum expectations.

Source: Institute for Defense Analyses Science and Technology Policy Institute, in press, b.

As Table 6 shows, there are wide variations across these six states in grade-level placement for these 27 topics. When the grade-level placement of the topics in these six highest-rated states is compared with the order of topics in Figure 7, there appears to be a relationship between the increasing complexity of the topics classified by TIMSS under numbers (Items 1, 2, 4, 10, 11, 12, 13, 14, 15, 18, 19, and 23) and under algebra (Items 5, 20, 25, and 26) and their introduction at increasingly higher grade levels. Nevertheless, Table 6 suggests that these six states (and probably most, if not all, of the others) spend a great deal of time in the primary grades on topics other than arithmetic (in the case of these six states, patterns, probability, and data analysis).

Patterns are labeled as an “algebra topic” in elementary and middle school mathematics curricula and assessments in this country; yet patterns are not emphasized in the curriculum of the high-performing countries in TIMSS; nor are patterns a topic of major importance in the Major Topics of School Algebra. The prominence given to patterns in K through 8 mathematics education in the U.S. is thus not supported by either the mathematical considerations or the data from TIMSS.

There is yet another striking difference with Figure 7. Not only do the A+ countries concentrate on arithmetic in the early grades, they also introduce geometry topics gradually from Grades 3 to 8, for the most part adding only one new geometry topic at each grade level. The seven geometry topics for the TIMSS high-performing countries in Figure 7 (Items 7, 8, 9, 16, 17, 21, and 22) first appear in Grades 3, 4, 4, 5, 6, 7, and 8, respectively. In contrast, in Table 6, in one or more of the highest-rated sets of state mathematics standards, geometry topics first appear in Grades 1, K/1, 3, 3, 3, 3, and 1 respectively. Indeed in these six states, there is little resemblance to the order of difficulty suggested in the composite curriculum of the A+ countries.

This comparison speaks to the reality of the spiral curriculum and the excessive number of topics taught at the elementary grades, even in the state frameworks that many consider as models for other states. Table 6 also helps make the point that a state’s mathematics standards, however highly their quality may be judged, do not necessarily correlate with student achievement in the state. These six states exhibit a wide range of student achievement on the 2007 NAEP mathematics tests for Grades 4 and 8. The quality of a state’s assessments and the extent to which its standards drive sound school curricula, as well as appropriate programs for teacher preparation and professional development, are intervening variables that strongly influence achievement. They may well override the quality of its standards.

C. Surveys of What College and Secondary Teachers See as Essential Concepts and Skills

1. Findings From the ACT Curriculum Survey

In 2007, ACT issued a report containing the results of its 2005–06 curriculum survey of a nationally representative sample of middle and high school teachers, high school counselors, and postsecondary regular and remedial course instructors in four major subjects (ACT, 2007). The survey indicates what instructors at postsecondary institutions believe is important and necessary for their entering students to know, and what middle and high school teachers are teaching. It therefore identifies the gap between postsecondary expectations and secondary school practice. For mathematics, ACT received responses from about 2,400 teachers, instructors, and counselors for an average response rate of 17%, with categorical response rates ranging from 11% (middle or junior high school teachers) to 26% (postsecondary instructors).

The responses of the instructors of postsecondary mathematics remedial courses were closer to the ratings of postsecondary mathematics entry-level-course instructors than to the ratings of high school mathematics teachers. On ratings of individual skills by mathematics strand on a 1 through 5 scale, postsecondary instructors for both regular and remedial mathematics courses rated the “4 skills that make up basic operations and applications” as most important (4.15) and the “12 skills that make up probability, statistics, and data analysis” as least important (1.76). On their rank ordering of eight mathematics strands in terms of importance, the postsecondary instructors rated basic operations and applications first and probability, statistics, and data analysis last. With respect to the course work needed for success in postsecondary mathematics, the content that both sets of postsecondary instructors rated the highest and that was being covered the least in instruction in arithmetic or in Algebra I courses was “solving quadratic equations and factoring (80%), working with rational exponents (41%), and using the quadratic formula (68%)” (ACT, 2007, p. 19).

2. Findings From the National Mathematics Advisory Panel Survey

As part of its deliberations in 2006, the Panel set as a priority a process to obtain information from a large national sample of teachers of introductory Algebra on, among other things, their views on their students’ mathematics preparation. The Panel developed a survey and gathered comments from 743 teachers (Hoffer, Venkataraman, Hedburg, & Shagle, 2007). About 28% of the Algebra I teachers were teaching at the middle or junior high school level, while almost all of the others were teaching in high schools.

The survey findings show that these algebra teachers generally describe their students’ backgrounds for Algebra I as weak. The two areas in which teachers report their students having the poorest preparation are 1) rational numbers and operations involving fractions and decimals and 2) solving word problems. The most frequent type of suggestion among the 578 teachers who responded in writing to an open-ended question was a greater focus in the elementary grades on proficiency with basic mathematics concepts and skills (Hoffer et al., 2007, p. 13). To a question on students using calculators in the early grades, those who wrote in a response specifically mentioned that they would like to see less use of calculators before students take their Algebra I class (p. 13). A number of teachers (n = 46) also mentioned student success in a “pre-Algebra” curriculum in the middle school as a requirement before they are allowed to take Algebra I (p. 14). In response to 10 options describing the challenges they face, a majority of the teachers (62%) saw “working with unmotivated students” as the “single most challenging aspect of teaching Algebra I successfully” (p. 23). The written-in responses, however, most frequently mentioned handling different skill levels in a single classroom (p. 23). In fact, a substantial number of teachers consider mixed-ability groupings to be a “moderate” (30%) or “serious” (23%) problem, an item with a combined rating of 53% for “moderate” and “serious” second only to the combined rating of 64% for “too little parent/family support” (p. 25). Finally, a majority of teachers favorably rated their Algebra I textbooks, with 90% agreeing or agreeing strongly that “the textbook includes the appropriate topics and content to teach the course” (Hoffer et al., 2007, Appendix D: Means and Confidence Intervals for Items in the National Survey of Algebra Teachers, p. 9-64).

D. Critical Foundations for Success in Algebra

The Task Group reviewed the concepts and skills indicated for teaching and learning in the elementary and middle grades 1) in the national curricula of the highest-performing countries on the TIMSS tests, 2) in the highest-rated U.S. state standards, 3) in NCTM’s *Focal Points*, 4) in the 2007 ACT curriculum survey, and 5) in the compiled ratings of the 743 algebra teachers surveyed for the Panel. The Task Group also took into consideration the structure of mathematics itself. Its structure requires teaching a sequence of major topics (from whole numbers to fractions, from positive numbers to negative numbers, and from the arithmetic of rational numbers to algebra) and an increasingly complex progression from explicit number computations to symbolic computations. The structural reasons for this sequence and its increasing complexity dictate what must be taught and learned before students take course work in Algebra. Based on all these considerations, the Task Group proposes the following three clusters of concepts and skills. The clusters reflect their judgment about what students need to learn thoroughly prior to algebra course work.

To prepare students for Algebra, the curriculum must simultaneously develop conceptual understanding, computational fluency, and problem-solving skills. These three aspects of learning are mutually reinforcing and should not be seen as competing for class time. The Critical Foundations identified and discussed below are not meant to comprise a complete preschool-to-algebra curriculum. However, the Task Group aims to recognize the Critical Foundations for the study of Algebra, whether as part of a dedicated algebra course in the seventh, eighth, or ninth grade, or within an integrated mathematics sequence in the middle and high school grades. These Critical Foundations deserve ample time in any mathematics curriculum. The foundations are presented in three distinct clusters of concepts and skills, each of which should incorporate the three aspects of learning noted here.

1. Fluency With Whole Numbers

By the end of the elementary grades, children should have a robust sense of number. This sense of number must include understanding place value, and the ability to compose and decompose whole numbers. It must clearly include a grasp of the meaning of the basic operations of addition, subtraction, multiplication, and division, including use of the commutative, associate, and distributive properties; the ability to perform these operations efficiently; and the knowledge of how to apply the operations to problem solving. Computational facility rests on the automatic recall of addition and related subtraction facts, and of multiplication and related division facts. It requires fluency with the standard algorithms for addition, subtraction, multiplication, and division. Fluent use of the algorithms not only depends on the automatic recall of number facts but also reinforces it. A strong sense of number also includes the ability to estimate the results of computations and thereby to estimate orders of magnitude, e.g., how many people fit into a stadium, or how many gallons of water are needed to fill a pool.

2. Fluency With Fractions

Before they begin algebra course work, middle school students should have a thorough understanding of positive as well as negative fractions. They should be able to locate both positive and negative fractions on the number line; represent and compare fractions, decimals, and related percents; and estimate their size. They need to know that sums, differences, products, and quotients (with nonzero denominators) of fractions are fractions, and they need to be able to carry out these operations confidently and efficiently. They should understand why and how (finite) decimal numbers are fractions and know the meaning of percentages. They should encounter fractions in problems in the many contexts in which they arise naturally, for example, to describe rates, proportionality, and probability. Beyond computational facility with specific numbers, the subject of fractions, when properly taught, introduces students to the use of symbolic notation and the concept of generality, both being an integral part of Algebra (Wu, 2001).

3. Particular Aspects of Geometry and Measurement

Middle-grade experience with similar triangles is most directly relevant for the study of algebra: Sound treatments of the slope of a straight line and of linear functions depend logically on the properties of similar triangles. Furthermore, students should be able to analyze the properties of two- and three-dimensional shapes using formulas to determine perimeter, area, volume, and surface area. They should also be able to find unknown lengths, angles, and areas.

E. Benchmarks for the Critical Foundations

In view of the sequential nature of mathematics, the Critical Foundations of Algebra described in the previous section require judicious placement in the grades leading up to Algebra. For this purpose, the Task Group suggests the following benchmarks as guideposts for state frameworks, for state assessments, and for school districts. There is no empirical research on the placement of these benchmarks, but they find justification in a comparison of national and international curricula. The benchmarks should be interpreted flexibly, to allow for the needs of students and teachers.

Fluency With Whole Numbers

1. By the end of Grade 3, students should be proficient with the addition and subtraction of whole numbers.

2. By the end of Grade 5, students should be proficient with multiplication and division of whole numbers.

Fluency With Fractions

1. By the end of Grade 4, students should be able to identify and represent fractions and decimals, and compare them on a number line or with other common representations of fractions and decimals.

2. By the end of Grade 5, students should be proficient with comparing fractions and decimals and common percents, and with the addition and subtraction of fractions and decimals.

3. By the end of Grade 6, students should be proficient with multiplication and division of fractions and decimals.

4. By the end of Grade 6, students should be proficient with all operations involving positive and negative integers.

5. By the end of Grade 7, students should be proficient with all operations involving positive and negative fractions.

6. By the end of Grade 7, students should be able to solve problems involving percent, ratio, and rate, and extend this work to proportionality.

Particular Aspects of Geometry and Measurement

1. By the end of Grade 5, students should be able to solve problems involving perimeter and area of triangles all quadrilaterals having at least one pair of parallel sides (i.e., trapezoids).

2. By the end of Grade 6, students should be able to analyze the properties of two-dimensional shapes and solve problems involving perimeter and area, and analyze the properties of three-dimensional shapes and solve problems involving surface area and volume.

3. By the end of Grade 7, students should be familiar with the relationship between similar triangles and the concept of the slope of a line.

VI. Does the Sequence of Mathematics Topics Prior to and During Algebra Course Work Affect Algebra Achievement?

As Schmidt et al. (2002) point out, the highest-performing countries in the TIMSS study employ somewhat differing curricula in the elementary grades. They conclude that a coherent, focused, and effective mathematics curriculum can be achieved in different ways. Over the past 15 years, many studies in the United States have examined the effects of recently developed mathematics curricula on student achievement at or across various grade levels. However, a search of the literature did not turn up any studies that sought to provide evidence on the effectiveness of these curricula (including their sequence of topics) for achievement in Algebra. A committee authorized by the National Academy of Sciences reviewed nearly 700 evaluations of 13 National Science Foundation-sponsored kindergarten through 12 mathematics curricula and 6 commercially developed mathematics curricula. Of these 700 evaluations, 147 met the committee's minimum criteria for scientific effectiveness and relevance, but they "did not permit one to determine the effectiveness of individual programs with a high degree of certainty, due to the restricted number of studies for any particular curriculum, limitations in the array of methods used, and the uneven quality of the studies" (Confrey & Stohl, 2004, p. 189).

There is no body of sound evidence showing particular multiyear mathematics curricula as more effective than others in preparing students in algebra course work. Thus, there is no basis in research for preferring a particular sequence.[14]

Beyond this central question are two related matters: a) whether an integrated approach or a single-subject sequence might be more effective for algebra course work and more advanced mathematics course work, and b) whether there are benefits to teaching the content of an Algebra I course before Grade 9. The next two subsections cover these issues.

A. Benefits of an Integrated or Single-Subject Approach For the Study of Algebra

The Task Group sought to examine if the differences in the way in which topics are sequenced in an integrated and single-subject approach for the study of Algebra lead to differences in algebra achievement. An integrated approach is defined as one in which the topics of high school mathematics are presented in some order other than the customary sequence in the United States of year-long courses in Algebra I, Geometry, Algebra II, and Precalculus.

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[14] It should be noted that the What Works Clearinghouse (WWC), managed by the U.S. Department of Education, reviews studies evaluating the effectiveness of current mathematics (and other) programs as part of an ongoing process, using standards that it formulated to rate the quality of the studies it reviews. WWC has found five middle school mathematics curricula supported by what it rated as high-quality research (see http://www.whatworks.ed.gov/). It is not clear how much weight should be attached to the ratings for these five curricula because for most of these curricula there are, so far, only one to three studies in all contributing to the ratings, and some of the studies contributing to the ratings did not find statistically significant results. None of these studies sought to determine the effectiveness of a particular multiyear mathematics curriculum implemented prior to formal algebra course work for success in Algebra.

The curricula of most higher-performing nations in the TIMSS study do not follow the single-subject sequence of Algebra I, Geometry, and Algebra II, but they also differ from the approach used in most U.S. integrated curricula. Instead, Algebra, Geometry, and Trigonometry are divided into blocks. The teaching of each block typically extends over several months and aims for mathematical closure. As a result, the need to revisit essentially the same material over several years, often referred to as "spiraling," is avoided. For an example, interested readers may consult the following Japanese textbooks for mathematics in Grades 9–11: The University of Chicago Mathematics Project, 1992; Kodaira, 1996, 1997.

The Task Group reviewed the most relevant studies to determine differences in achievement between curricular approaches that differentially shape the sequence of topics prior to or during the study of Algebra. They uncovered no research that clearly compared the use of textbooks featuring an integrated approach to the use of textbooks reflecting more of a single-subject approach. Nor did the researchers at the GE Foundation's Urban Institute, who found 156 comparison studies of 18 middle and high school mathematics curricula that met their criteria for inclusion (Clewell et al., 2004). Their criteria included 1) an experimental or quasi-experimental design, 2) comparison groups, and 3) measures of student achievement that included but were not limited to test scores. Although the researchers found that students using six of the curricula scored higher than comparison students on both a majority of the standardized, state tests used, or both, and on a majority of the curriculum-based tests used, the reviewers stress that *only* 3 of the 156 studies they examined provided details on what was in the comparison curricula. Thus, it is not clear with what these six significantly more effective curricula were being compared in the other 153 studies, and there is no clear body of research from which conclusions on this question can be drawn.

There is no basis in research for preferring either a single-subject sequence or an integrated sequence for the teaching of school mathematics at the level of Algebra or above.

Although there seems to be little descriptive material on the differences in mathematical content between integrated and single-subject approaches for high school algebra, the differences may be striking. STPI compared North Carolina's 2003 standards for an integrated approach to high school mathematics with the state's standards for the single-subject sequence in high school (North Carolina State Board of Education, 2003; Institute for Defense Analyses Science and Technology Policy Institute, in press, c). It found that the integrated mathematics sequence for Grades 9 through 12 includes 1) all of the course objectives for the Algebra I course, 2) 7 out of 8 of the course objectives for the geometry course, and 3) 9 of the 15 course objectives for the Algebra II course. The comparison determined that North Carolina students completing *4 years* of the integrated mathematics sequence would not complete all the course objectives addressed by students in the *3-year* Algebra I, Geometry, and Algebra II sequence. In this case, at least, high school students enrolled in mathematics courses using textbooks featuring an integrated approach may not be in a position to take more advanced mathematics course work in their senior year, as can high school students at present who are able to enroll in an Algebra II course in their sophomore or junior year.

B. Research on the Timing of Algebra Course Work

The final question the Task Group addressed concerns the benefits of teaching the content of an Algebra I course before Grade 9, that is, of giving students an opportunity to learn more mathematics before Grade 9 than is expected in many, if not most, state curriculum frameworks. This specific question is relevant to the Task Group's broader question of the math necessary prior to algebra because the content of Algebra I is provided to some or all students in Grade 8 or even Grade 7 in many other countries (as well as to many students in this country), and the specific grade at which it is offered may well affect the grade-by-grade sequence of topics in a school's curriculum prior to algebra course work. The Task Group does not indicate specific grade levels for teaching or learning the essential concepts and skills in the three clusters proposed in the previous section, to allow schools the option of working out a coherent sequence of these concepts and skills at differing elementary grade levels, depending on when they choose to provide the content of an Algebra I course.

According to information gathered for the 2005 Grade 8 NAEP tests, about 39% of U.S. students have completed a 1- or 2-year Algebra I course in Grade 8 or have taken Algebra I in Grade 7.[15] Although clear and current international data across a wide range of countries on the timing of algebra course work cannot be located, it is clear from TIMSS data and the work of Schmidt et al. (2002) that students in the A+ countries study Algebra as well as Geometry in Grades 7 and 8. In contrast, in a large number of U.S. schools, an algebra course is not available in those grades. Schmidt et al. determined that "while 80% of eighth-graders had access to a 'regular' math course, only 66.5% of eighth-graders attend schools that even offer an algebra course. That is, a full third of eighth-graders don't even have such a course as an option" (p. 14).

Yet, a report from the U.S. Department of Education (1997) articulated the need to "provide all students the opportunity to take Algebra I or a similarly demanding course that includes fundamental algebraic concepts in the 8th grade and more advanced math and science courses in all four years of high school." It urged schools to "build the groundwork for success in algebra by providing a rigorous curriculum in grades K–7 that moves beyond arithmetic and prepares students for the transition to algebra."

The U.S. Department of Education report (1997) was the background for the Task Group's interest in finding research evidence on the long-term benefits of completing Algebra I before Grade 9. A search of the literature produced six studies that met the Panel's design criteria and included Algebra or mathematics achievement as an outcome (Jones, Davenport, Bryson, Bekhuis, & Zwick, 1986; Lee, Burkam, Chow-Hoy, Smerdon, & Geverdt, 1998; Ma, 2000, 2005; Smith, 1996; and Wilkins & Ma, 2002). Smith's study used algebra achievement as an outcome, but the others used general tests that measured student performance on a variety of mathematical concepts and skills. All of the studies were analyses of large national data sets [High School and Beyond (HS&B), Longitudinal Survey of American Youth (LSAY), National Education Longitudinal Study (NELS): 88, and the High School Effectiveness Study] and all examined the relationships between high school mathematics achievement and

[15] See http://nces.ed.gov/nationsreportcard/nde/viewresults.asp.

students' course-taking patterns in mathematics. Because students are never randomly assigned to specific course-taking patterns in mathematics in any school, one cannot definitively determine whether student achievement is the result of the courses that students take, whether their course-taking patterns result from their achievement, or if both their course-taking patterns and their achievement are the result of other, unmeasured factors. Nevertheless, these six studies are informative and appear to have used rigorous methods to analyze the course-taking patterns in mathematics that students chose. They all controlled for important school and student characteristics, including prior achievement, although they did not control for exactly the same variables, and some studies controlled for more variables than others.

Four of the six studies highlight the relationship between the timing of Algebra I and mathematics achievement (Ma, 2000, 2005; Smith, 1996; and Wilkins & Ma, 2002). Three of the four (two of which used the same LSAY data set) found that students who took Algebra prior to starting high school tended to have higher mathematics achievement in high school (Ma, 2000, 2005; Smith) than those who did not. Ma (2000) examined the effect of course taking in one year on achievement in the following year. Controlling for socioeconomic status (SES), gender, age, and prior mathematics achievement, he found that students who took Pre-Algebra or Algebra I in Grade 7 had higher average mathematics achievement in Grade 8 than those who did not take these courses in Grade 7; in addition, taking Pre-Algebra in Grade 7 had a greater effect on Grade 8 achievement than taking Algebra I in Grade 7. Those who took Pre-Algebra, Algebra I, Algebra I Honors, or Geometry in Grade 8 had higher average mathematics achievement in Grade 9 than those who did not take one of these courses; in addition, boys did better than girls. Of these four courses, taking Algebra I in Grade 8 had the largest impact on Grade 9 achievement, followed by Pre-Algebra, Algebra I Honors, then Geometry. However, the mathematics courses that students took in Grade 9 did not predict Grade 10 achievement.

Smith's study (1996) compared students who took Algebra prior to starting high school with those who took it at the beginning of high school (Grade 9). She, too, found that students who took Algebra early had higher mathematics achievement scores in Grade 12, even after controlling for background characteristics and mathematics achievement in Grade 10. Smith also found that students who took Algebra early, on average, took more advanced mathematics courses in high school.

The final two studies (Jones et al., 1986; Lee et al., 1998) also highlight the relationship between the number of mathematics courses taken in high school and students' mathematics achievement. Controlling for prior mathematics achievement, verbal ability, and SES, Jones et al. found that, on average, students who took a larger number of advanced mathematics courses in high school (Algebra I or higher) had higher mathematics achievement in Grade 12 than students who took fewer courses. Conversely, Lee et al. found that, after controlling for characteristics of students and schools, students who took more low-level courses (lower than Algebra I), on average, had low mathematics achievement scores in Grade 12 and reached lower levels of mathematics course work by Grade 12 than students who took more high-level courses. Together, findings from these three studies (Smith, 1996; Jones et al.; and Lee et al.) suggest that students who take more mathematics courses at the level of Algebra 1 or higher have, on average, higher mathematics achievement in high school than students who take fewer courses, controlling for background characteristics.

It is important to note that these six studies drew on four national data sets. Three analyzed LSAY data (Ma, 2000, 2005; Wilkins & Ma, 2002), two used HS&B data (Smith, 1996; Jones et al., 1986), and one used data from NELS: 88 and the High School Effectiveness Study (Lee et al., 1998). The consistency of their findings is striking. The studies by Ma and others provide some evidence that there are long-term benefits for Grade 7 or 8 students with the requisite mathematical background for algebra if they can take an authentic Algebra course in Grade 7 or 8: higher mathematics achievement in high school and the opportunity to take advanced mathematics course work in Grade 11 or 12.

If students have the opportunity to take Algebra I in Grade 7 or 8, they will be able to enroll in precalculus, calculus, or other advanced mathematics courses in Grade 11 or 12. The importance of being able to take a calculus course before graduation, or at the least a precalculus course (or a post-Algebra II course that includes trigonometry), is underscored by a 2003 survey of admission requirements for Massachusetts public and private institutions of higher education offering 4-year engineering programs (Massachusetts Department of Education, 2003).

In sum, there is no research demonstrating that a specific multigrade sequence of mathematics topics assures success in Algebra. Nor is there a body of research from which one may draw conclusions about the relative effectiveness of either an integrated or a single-subject approach to the study of Algebra and more advanced mathematics. However, research evidence, as well as the experience of other countries, supports the value of preparing a higher percentage of students than the U.S. does at present to complete an Algebra I course or its equivalent by Grade 7 or 8, and of providing such course work in Grade 7 or 8.

VII. Recommendations

This Task Group affirms that Algebra is the gateway to more advanced mathematics and to most postsecondary education. All schools and teachers of mathematics must concentrate on providing a solid mathematics education to all elementary and middle school students so that all of them can enroll and succeed in Algebra. Students need to be soundly prepared for Algebra and then well taught in Algebra, regardless of the grade level at which they study it. To improve the teaching of Algebra, the Task Group proposes the following eight recommendations:

1) The Task Group recommends that school algebra be consistently understood in terms of the Major Topics of School Algebra given in this report on page 5.

2) The Major Topics of School Algebra, accompanied by a thorough elucidation of the mathematical connections among these topics, should be the main focus of Algebra I and Algebra II standards in state curriculum frameworks, in Algebra I and Algebra II courses, in textbooks for these two levels of Algebra whether for integrated curricula or otherwise, and in end-of-course assessments of these two levels of Algebra. The Task Group also recommends use of the Major Topics of School Algebra in revisions of mathematics standards at the high school level in state

curriculum frameworks, in high school textbooks organized by an integrated approach, and in grade-level state assessments using an integrated approach at the high school level, by Grade 11 at the latest.

3) Proficiency with whole numbers, fractions, and particular aspects of geometry and measurement are the Critical Foundations of Algebra (p. 40). Emphasis on these essential concepts and skills must be provided at the elementary- and middle-grade levels. The coherence and sequential nature of mathematics dictate the foundational skills that are necessary for the learning of algebra. By the nature of algebra, the most important foundational skill is proficiency with fractions (including decimals, percent, and negative fractions). The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in Algebra can be expected.

4) The Benchmarks proposed by the Task Group on page 42 should be used to guide classroom curricula, mathematics instruction, and state assessments. They should be interpreted flexibly, to allow for the needs of students and teachers.

5) International studies show that high-achieving nations teach for proficiency in a few topics, in comparison with the U.S. mile-wide-inch-deep curriculum. A coherent progression, with an emphasis on proficiency in key topics, should become the norm in elementary and middle school mathematics curricula. What should be avoided in mathematics is an approach that continually revisits topics year after year without closure.

6) All school districts should ensure that all prepared students have access to an authentic algebra course—and should prepare more students than at present to enroll in such a course by Grade 8. The word “authentic” is used here as a descriptor of a course that addresses algebra consistently with the Major Topics of School Algebra. Students must be prepared with the mathematical prerequisites for this course according to the Critical Foundations and the Benchmarks.

7) Publishers must ensure the mathematical accuracy of their materials. Those involved with developing mathematics textbooks and related instructional materials need to engage mathematicians, as well as mathematics educators, in writing, editing, and reviewing these materials.

8) Teacher education programs and licensure tests for early childhood teachers, including all special education teachers at this level, should fully address the topics on whole numbers, fractions, and the appropriate geometry and measurement topics in the Critical Foundations of Algebra, as well as the concepts and skills leading to them; for elementary teachers, including elementary-level special education teachers, all topics in the Critical Foundations of Algebra and those topics typically covered in an introductory Algebra course; and for middle school teachers, including middle school special education teachers, the Critical Foundations of Algebra and all of the Major Topics of School Algebra.

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APPENDIX A

Table A-1: Comparison of the Major Algebra Topics in Five Sets of Algebra I and Algebra II Textbooks With the List of Major Topics of School Algebra

Major Topics of School Algebra		Glencoe McGraw-Hill 2008	Glencoe McGraw-Hill: Calif. Edition 2005	Prentice Hall 2007	Holt, Rinehart and Winston 2007	Houghton Mifflin 1988
Symbols and Expressions	Polynomial Expressions	Variables and Expressions	Variables and Expressions	Using Variables	Variables and Expressions	Variables and Equations
		Writing Equations	Expressions and Formulas	Properties of Numbers	Simplifying Expressions	
		Solving Equations by Using Addition and Subtraction	Commutative and Associative Properties	Adding Rational Numbers	Order of Operations	
		Expressions and Formulas	The Distributive Property	Subtracting Rational Numbers	Adding and Subtracting Polynomials	
		Order of Operations	Adding and Subtracting Rational Numbers	Multiplying and Dividing Rational Numbers		
		The Distributive Property	Multiplying Rational Numbers	Exponents and Order of Operations		
		Commutative and Associative Properties	Dividing Rational Numbers	The Distributive Property		
		Multiplying Monomials	Operations With Polynomials	Adding and Subtracting Polynomials		
		Adding and Subtracting Polynomials				
		Identity and Equality Properties	Order of Operations			
	Rational Expressions	Dividing Monomials	Rational Expressions	Solving Rational Equations	Simplifying Algebraic Expressions	Algebraic Fractions
		Rational Expressions	Multiplying Rational Expressions	Algebraic Expressions	Simplifying Rational Expressions	Adding and Subtracting Fractions
		Multiplying Rational Expressions	Dividing Rational Expressions	Rational Expressions	Multiplying and Dividing Rational Expressions	Working With Rational Expressions
		Rational Expressions With Like Denominators	Dividing Polynomials	Adding and Subtracting Rational Expressions	Adding and Subtracting Rational Expressions	Fractional Equations
		Rational Expressions With Unlike Denominators	Rational Expressions With Like Denominators	Solving Rational Equations	Solving Rational Equations	Polynomial Long Division
		Rational Equations and Functions	Rational Expressions With Unlike Denominators	Dividing Polynomials	Multiplying and Dividing Rational Expressions	Mixed Expressions
		Dividing Rational Expressions	Mixed Expressions and Complex Fractions	Simplifying Rational Expressions	Adding and Subtracting Rational Expressions	Rational Algebraic Expressions
		Mixed Expressions and Complex Fractions	Solving Rational Equations	Multiplying and Dividing Rational Expressions	Adding and Subtracting Polynomials	
		Multiplying and Dividing Rational Expressions	Solving Rational Equations and Inequalities			
		Adding and Subtracting Rational Expressions	Multiplying and Dividing Rational Expressions			
		Dividing Polynomials	Adding and Subtracting Rational Expressions			
			Operations With Polynomials			

Continued on p. 3-54

Table A-1, continued

Major Topics of School Algebra		Glencoe McGraw-Hill 2008	Glencoe McGraw-Hill: Calif. Edition 2005	Prentice Hall 2007	Holt, Rinehart and Winston 2007	Houghton Mifflin 1988
Symbols and Expressions	Arithmetic and Geometric Sequences and Series	Arithmetic Sequences	Arithmetic Sequences	Arithmetic Sequences	Arithmetic Sequences	Arithmetic and Geometric Series
		Arithmetic Series	Geometric Sequences	Geometric Sequences	Geometric Sequences	Sequences
		Geometric Series	Arithmetic Series	Geometric Series	Introduction to Sequences	
		Infinite Geometric Series	Infinite Geometric Series	Arithmetic Series	Series and Summation Notation	
		Recursion and Special Sequences	Recursion and Special Sequences		Arithmetic Sequences and Series	
		Geometric Sequences	Geometric Series		Geometric Sequences and Series	
					Mathematical Induction and Infinite Geometric Series	
Linear Equations	Real Numbers as Points on the Number Line	Properties of Real Numbers	Rational Numbers on the Number Line	Exploring Real Numbers	Adding and Subtracting Real Numbers	Numbers on a line
			Properties of Real Numbers	Properties of Real Numbers	Multiplying and Dividing Real Numbers	Operating With Real Numbers
					Properties of Real Numbers	Rational Numbers
						Irrational Numbers
						Working With Real Numbers: Addition and Subtraction
						Dividing Real Numbers
	Linear Equations and Their Graphs	Solving Equations With the Variable on Each Side	Writing Equations in Slope-Intercept Form	Point-Slope Form and Writing Linear Equations	The Slope Formula	Linear Equations
		Geometry: Parallel and Perpendicular Lines	Writing Equations	Slope-Intercept Form	Slope-Intercept Form	Linear Equations and Their Graphs
		Graphing Equations in Slope-Intercept Form	Identity and Equality Properties	Parallel and Perpendicular Lines	Rate of Change and Slope	
		Solving for a Specific Variable	Relations	Linear Equations	Point-Slope Form	Solving Equations and Solving Problems
		Solving Equations by Using Multiplication and Division	Solving Equations by Using Addition and Subtraction	Solving Equations	Using Intercepts	Transforming Equations: Addition and Subtraction
		Solving Multistep Equations	Geometry: Parallel and Perpendicular Lines	Equations With Variables on Both Sides	Slopes of Parallel and Perpendicular Lines	Transforming Equations: Multiplication and Division
		Linear Equations	Graphing Linear Equations	Solving Two-Step Equations	Solving Linear Equations and Inequalities	Slope of a Line
		Writing Linear Equations	Slope-Intercept Form	Solving Multistep Equations	Solving for a Variable	Slope-Intercept Form of a Linear Equation
		Writing Point-Slope Form	Solving Equations	Equations and Problem Solving	Solving Equations by Adding or Subtracting	Determining an Equation of a Line
		Similar Triangles	Solving Equations by Using Multiplication and Division	Proportions and Similar Figures	Solving Equations by Multiplying or Dividing	
		Rate of Change and Slope	Solving Multistep Equations		Solving Two-Step and Multistep Equations	
		Writing Equations in Slope-Intercept Form	Solving Equations With the Variable on Each Side		Solving Equations With Variables on Both Sides	

Continued on p. 3-55

Table A-1, continued

Major Topics of School Algebra		Glencoe McGraw-Hill 2008	Glencoe McGraw-Hill: Calif. Edition 2005	Prentice Hall 2007	Holt, Rinehart and Winston 2007	Houghton Mifflin 1988
Linear Equations	Linear Equations and their Graphs	Slope	Solving Equations and Formulas		Square Roots and Real Numbers	
		Linear Graphs	Equations as Relations			
			Writing Equations in Point-Slope Form			
			Linear Equations			
			Similar Triangles			
			Slope			
			Writing Linear Equations			
	Solving Problems With Linear Equations	Weighted Averages	Percent of Change	Percent of Change	Applications of Proportions	Ratio and Proportion
		Weighted Averages	Weighted Averages	Rate of Change and Slope		Percent Problems
						Variation and Proportion
		Percent of Change				Mixture and Work Problems
	Linear Inequalities and Their Graphs	Solving Compound Inequalities	Solving Inequalities by Addition and Subtraction	Solving Inequalities Using Multiplication and Division	Solving Linear Inequalities	Inequalities in One Variable
		Graphing Inequalities in Two Variables	Solving Inequalities by Multiplication and Division	Inequalities and Their Graphs	Graphing and Writing Inequalities	Inequalities in Two Variables
		Solving Inequalities	Solving Multistep Inequalities	Solving Inequalities Using Addition and Subtraction	Solving Inequalities by Adding or Subtracting	Working With Inequalities
		Graphing Inequalities	Solving Compound Inequalities	Linear Inequalities	Solving Inequalities by Multiplying or Dividing	Working With Absolute Value
		Solving Equations	Graphing Inequalities in Two Variables	Solving Multistep Inequalities	Solving Two-Step and Multistep Inequalities	Solving Inequalities
		Solving Inequalities by Addition and Subtraction	Solving Inequalities	Compound Inequalities	Solving Inequalities With Variables on Both Sides	Graphing Linear Inequalities
		Solving Inequalities by Multiplication and Division	Graphing Inequalities	Two-Variable Inequalities	Solving Compound Inequalities	Solving Systems of Linear Inequalities.
		Solving Inequalities Involving Absolute Value	Graphing Systems of Inequalities	Solving Inequalities	Linear Inequalities in Two Variables	
		Solving Multistep Inequalities	Linear Programming	Absolute Value Equations and Inequalities	Solving Systems of Linear Inequalities	
		Solving Compound and Absolute Value Inequalities	Solving Systems of Inequalities by Graphing	Linear Programming	Linear Programming	
		Linear Programming		Systems of Inequalities		
		Solving Systems of Inequalities by Graphing				
		Graphing Systems of Inequalities				
	Graphing and Solving Systems of Simultaneous Linear Equations	Elimination Using Addition and Subtraction	Graphing Systems of Equations	Solving Systems by Graphing	Solving Systems by Graphing	Using Two Variables
		Solving Systems of Equations Algebraically	Substitution	Solving Systems Using Substitution	Solving Systems by Substitution	Solving Systems of Linear Equations
		Graphing Systems of Equations	Elimination Using Addition and Subtraction	Solving Systems Using Elimination	Solving Systems by Elimination	Linear Systems

Continued on p. 3-56

Table A-1, continued

Major Topics of School Algebra		Glencoe McGraw-Hill 2008	Glencoe McGraw-Hill: Calif. Edition 2005	Prentice Hall 2007	Holt, Rinehart and Winston 2007	Houghton Mifflin 1988
Linear Equations	Graphing and Solving Systems of Simultaneous Linear Equations	Solving Systems of Equations in Three Variables	Elimination Using Multiplication	Applications of Linear Systems	Solving Special Systems	Systems of Equations
		Applying Systems of Linear Equations	Solving Systems of Equations by Graphing	Systems of Linear Inequalities	Using Graphs and Tables to Solve Linear Systems	Systems of Linear Equations in Three Variables
		Elimination Using Multiplication	Solving Systems of Equations Algebraically	Graphing Systems of Equations	Using Algebraic Methods to Solve Linear Systems	
		Solving Systems of Equations by Graphing	Solving Systems of Equations in Three Variables	Solving Systems Algebraically	Solving Linear Systems in Three Variables	
				Systems With Three Variables	Linear Equations in Three Dimensions	
Quadratic Equations	Factors and Factoring of Quadratic Polynomials with Integer Coefficients	Substitution	Factors and Greatest Common Factors	Solving Quadratic Equations	Factors and Greatest Common Factors	Factoring Integers
		Monomials and Factoring	Factoring Using the Distributive Property	Factoring to Solve Quadratic Equations	Factoring by Greatest Common Factors	Monomial Factors of Polynomials
		Factoring Trinomials: x^2+bx+c	Factoring Trinomials: x^2+bx+c	Multiplying and Factoring	Factoring x^2+bx+c	Multiplying Binomials Mentally
		Factoring Trinomials: ax^2+bx+c	Factoring Trinomials: ax^2+bx+c	Multiplying Binomials	Factoring ax^2+bx+c	Differences of Squares
		Factoring Using the Distributive Property	Factoring Differences of Squares	Factoring Trinomials of the Type x^2+bx+c	Factoring Special Products	Squares of Binomials
		Solving Quadratic Equations by Factoring	Perfect Squares and Factoring	Factoring Trinomials of the Type ax^2+bx+c	Choosing a Factoring Method	Factoring Pattern for x^2+bx+c, c positive
		Factoring Differences of Squares	Solving Quadratic Equations by Graphing	Factoring by Grouping	Solving Quadratic Equations by Graphing and Factoring	Factoring Pattern for x^2+bx+c, c negative
		Perfect Squares and Factoring	Solving Quadratic Equations by Factoring		Solving Quadratic Equations by Factoring	Factoring Pattern for ax^2+bx+c
						Factoring by Grouping
					Solving Quadratic Equations by Using Square Roots	Using More Than One Method of Factoring
						Solving Equations by Factoring
						Solving Problems by Factoring
						Quadratic Equations With Perfect Squares
	Completing the Square in Quadratic Expressions	Completing the Square	Solving Quadratic Equations by Completing the Square	Completing the Square	Completing the Square	Completing the Square
		Solving Quadratic Equations by Completing the Square	Completing the Square	Factoring Quadratic Expressions		
				Completing the Square		

Continued on p. 3-57

Table A-1, continued

Major Topics of School Algebra		Glencoe McGraw-Hill 2008	Glencoe McGraw-Hill: Calif. Edition 2005	Prentice Hall 2007	Holt, Rinehart and Winston 2007	Houghton Mifflin 1988
Quadratic Equations	Quadratic Formula and Factoring of General Quadratic Polynomials	The Quadratic Formula and the Discriminant	The Quadratic Formula and the Discriminant	The Quadratic Formula	The Quadratic Formula and the Discriminant	Quotients and Factoring
				Using the Discriminant	The Quadratic Formula	Products and Factors
						The Quadratic Formula
						The Quadratic Formula and the Discriminant
	Using the Quadratic Formula to Solve Equations	Solving Quadratic Equations by Using the Quadratic Formula	Solving Quadratic Equations by Using the Quadratic Formula	Not Available	Not Available	Solving Quadratic Equations
			Solving Equations Using Quadratic Techniques			Quadratic Equations
						Roots of Quadratic Equations
						Using Quadratic Equations
Functions	Linear Functions	Representing Relations	Graphs and Functions	Patterns and Functions	Identifying Linear Functions	Functions and Relations
		Proportional and Nonproportional Relationships	Ratios and Proportions	Ratio and Proportion	Transforming Linear Functions	Functions
		Relations and Functions	Functions	Relations and Functions	Writing Function	Functions and Relations
		Functions and Graphs	Writing Equations from Patterns	Relating Graphs to Events	Introduction to Parent Functions	Functions Defined by Equations
		Linear Functions	Classes of Functions	Functions Rules, Tables, and Graphs	Graphing Relationships	Functions Defined by Tables and Graphs
		Operations on Functions	Relations and Functions	Writing a Function Rule	Relations and Functions	Direct Variation
		Representing Functions	Operations on Functions	Describing Number Patterns	Introduction to Function	
		Direct, Joint, and Inverse Variation	Slope and Direct Variation	Relations and Functions	Relations and Functions	Direct and Inverse Variation Involving Squares
				Families of Functions	Rates, Ratios, and Proportions	Joint and Combined Variation
				Mathematical Patterns	Graphing Linear Functions	
				Direct Variation	Writing Linear Functions	
				Applying Linear Functions	Function Notations	
				Standard Form	Direct Variation	
					Variation Functions	
					Operations With Functions	
					Multiple Representations of Functions	
					Transforming Quadratic Functions	
					Using Transformations to Graph Quadratic Functions	
					Transforming Linear Functions	
					Proportional Reasoning	

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Table A-1, continued

Major Topics of School Algebra		Glencoe McGraw-Hill 2008	Glencoe McGraw-Hill: Calif. Edition 2005	Prentice Hall 2007	Holt, Rinehart and Winston 2007	Houghton Mifflin 1988
Functions	**Quadratic Functions – Word Problems Involving Quadratic Functions**	Not Available	Not Available	Quadratic Functions	Properties of Quadratic Functions in Standard Form	Not Available
				Quadratic Functions	Characteristics of Quadratic Functions	
	Graphs of Quadratic Functions	Graphing Quadratic Functions	Circles	Properties of Parabolas	Graphing Functions	Quadratic Functions and Their Graphs
		Parabolas	Analyzing Graphs of Quadratic Functions	Transforming Parabolas	Parabolas	Conic Sections: Circles and Parabolas
		Analyzing Graphs of Quadratic Functions	Parabolas	Parabolas	Graphing Quadratic Functions	Linear and Quadratic Functions
		Circles	Graphing Quadratic Functions	Circles	Solving Quadratic Equations by Graphing	
		Solving Quadratic Equations by Graphing	Solving Quadratic Equations by Graphing	Exploring Quadratic Graphs	Circles	
		Graphing Quadratic Functions	Graphing Quadratic Functions		Identifying Quadratic Functions	
			Classes of Functions			
		Solving Quadratic Equations by Graphing				
	Polynomial Functions (including graphs of basic functions)	Analyzing Graphs of Polynomials Functions	Polynomial Functions	Polynomial Functions	Investigating Graphs of Polynomial Functions	Products of Polynomials
		Polynomial Functions	Analyzing Graphs of Polynomial Functions		Transforming Polynomial Functions	Polynomial Division
			Graphing Polynomial Functions			
	Simple Nonlinear Functions Simple Nonlinear Functions (e.g., square and cube root functions; absolute value; rational functions; step functions)		Square Roots and Real Numbers	Graphing Rational Functions	Transforming Functions	Applying Fractional Equations
		Inverse Variation		Finding and Estimating Square Roots	Rational Functions	Inverse Functions and Equations
		Special Functions	Graphing Rational Functions	Graphing Absolute Value Equations	Solving Absolute-Value Equations and Inequalities	The Reciprocal of a Number
		Square Root Functions and Inequalities	Solving Absolute Value Equations	Absolute Value Equations and Inequalities	Absolute-Value Functions	Inverse Variation
		nth Roots	Solving Compound and Absolute Value Inequalities	Absolute Value Functions and Graphs	Rational Functions	
		Solving Absolute Value Equations	Square Root Functions and Inequalities	Inverse Relations and Functions	Solving Rational Equations and Inequalities	
		Inverse Functions and Relations	Direct, Joint, and Inverse Variation	Inverse Variation	Radical Functions	
		Direct, Joint, and Inverse Variation	Inverse Functions and Relations	Rational Functions and Their Graphs	Solving Radical Equations and Inequalities	
		Solving Rational Equations and Inequalities	Classes of Functions	The Reciprocal Function Family	Inverse Variation	

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Table A-1, continued

Major Topics of School Algebra		Glencoe McGraw-Hill 2008	Glencoe McGraw-Hill: Calif. Edition 2005	Prentice Hall 2007	Holt, Rinehart and Winston 2007	Houghton Mifflin 1988
Functions	**Simple Nonlinear Functions Simple Nonlinear Functions (e.g., square and cube root functions; absolute value; rational functions; step functions)**	Graphing Rational Functions	Inverse Variation	Graphing Square Root and Other Radical Functions	Inverses of relations and Functions	
				Graphing Square Root Functions	Piecewise Functions	
					Functions and Their Inverses	
					Square-Root Functions	
					Solving Nonlinear Systems	
					Square Roots	
	Rational Exponents, Radical Expressions, and Exponential Functions	Exponential Growth and Decay	Exponential Functions	Zero and Negative Exponents	Integer Exponents	Radical Expressions
		Rational Exponents	Growth and Decay	Multiplication Properties of Exponents	Multiplication Properties of Exponents	Problems Involving Exponents
		Simplifying Radical Expressions	Simplifying Radical Expressions	More Multiplication Properties of Exponents	Division Properties of Exponents	Using the Laws of Exponents
		Operations With Radical Expressions	Operations With Radical Expressions	Division Properties of Exponents	Radical Expressions	Exponential Functions
		Growth and Decay	Radical Equations	Exponential Functions	Adding and Subtracting Radical Expressions	Negative Exponents
		Properties of Exponents	Radical Expressions	Exponential Growth and Decay	Multiplying and Dividing Radical Expressions	Roots and Radicals
		Exponential Functions	Exponential Growth and Decay	Simplifying Radicals	Powers and Exponents	Powers of Monomials
		Operations With Radical Expressions	Exponential Functions	Operations With Radical Expressions	Exponential Functions	Rational exponents
		Solving Radical Equations and Inequalities	Rational Exponents	Solving Radical Equations	Exponential Growth and Decay	Exponential Growth and Decay
		Classes of Functions	Radical Equations and Inequalities	Rational Exponents	Linear, Quadratic, and Exponential Models	Roots of Real Numbers
		Radical Equations	Properties of Exponents	Properties of Exponential Functions	Solving Radical Equations	
		Exponential Functions	Roots of Real Numbers	Multiplying and Dividing Radical Expressions	Exponential Functions, Growth, and Decay	
			nth Roots	Binomial Radical Expressions	Radical Expressions and Rational Exponents	
				Solving Square Root and Other Radical Equations	Properties of Exponents	
				Roots and Radical Expressions	Rational Exponents	
				Choosing a Linear, Quadratic, or Exponential Model		
				Function Operations		
	Logarithmic Functions	Logarithms and Logarithmic Functions	Logarithms and Logarithmic Functions	Logarithmic Functions as Inverses	Logarithmic Functions	Logarithmic Functions
		Properties of Logarithms	Properties of Logarithms	Properties of Logarithms	Properties of Logarithms	The Natural Logarithm Function

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Table A-1, continued

Major Topics of School Algebra		Glencoe McGraw-Hill 2008	Glencoe McGraw-Hill: Calif. Edition 2005	Prentice Hall 2007	Holt, Rinehart and Winston 2007	Houghton Mifflin 1988
Functions	Logarithmic Functions	Common Logarithms	Common Logarithms	Exponential and Logarithmic Equations	Exponential and Logarithmic Equations and Inequalities	
		Base *e* and Natural Logarithms	Base *e* and Natural Logarithms	Natural Logarithms	The Natural Base, *e*	
					Transforming Exponential and Logarithmic Functions	
	Trigonometric Functions	Graphing Trigonometric Functions	Trigonometric Ratios	Trigonometric Ratios	Trigonometric Ratios	Trigonometric Functions
		Trigonometric Functions of General Angles	Graphing Trigonometric Functions	Angles of Elevation and Depression	Graphs of Sine and Cosine	Triangle Trigonometry
		Circular Functions	Translations of Trigonometric Graphs	Solving Trigonometric Equations Using Inverses	Graphs of Other Trigonometric Functions	Circular Functions and Their Graphs
		Inverse Trigonometric Functions	Circular Functions	Right Triangle and Trigonometric Ratios	Solving Trigonometric Equations	
		Translations of Trigonometric Graphs	Inverse Trigonometric Functions	Radian Measure	Inverses of Trigonometric Functions	
		Solving Trigonometric Equations	Solving Trigonometric Equations	The Tangent Function		
		Verify Trigonometric Functions		The Sine Function		
				The Cosine Function		
				Translating Sine and Cosine Function		
				Exploring Periodic Data		
				Reciprocal Trigonometric Functions		
	Fitting Simple Mathematical Models to Data	Scatter Plots and Lines of Fit	Modeling “Real-World” Data: Using Scatter Plots	Scatter Plots	Scatter Plots and Trend Lines	
			Statistics: Displaying and Analyzing Data	Scatter Plots and Equations of Lines	Curve Fitting With Linear Models	
			Statistics: Analyzing Data by Using Tables and Graphs	Using Linear Models	Modeling “Real-World” Data	
			Statistics: Scatter Plots and Lines of Fit			
			Statistics: Using Scatter Plots			
	Roots and Factorization of Polynomial Forms	The Remainder and Factor Theorems	Factoring Polynomials	Theorems About Roots of Polynomial Equations	Factoring Polynomials	Factors of Polynomials
		Polynomials	The Remainder and Factor Theorems	Polynomials and Linear Factors	Polynomials	Theory of Polynomial Equations
		Adding and Subtracting Polynomials	Multiplying Monomials	Multiplying Special Cases	Adding and Subtracting Polynomials	Solving Polynomial Equation
		Multiplying a Polynomial by a Monomial	Dividing Monomials	Dividing Polynomials	Special Products of Binomials	Adding and Subtracting Polynomials

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Table A-1, continued

Major Topics of School Algebra		Glencoe McGraw-Hill 2008	Glencoe McGraw-Hill: Calif. Edition 2005	Prentice Hall 2007	Holt, Rinehart and Winston 2007	Houghton Mifflin 1988
Functions	**Roots and Factorization of Polynomial Forms**	Operations with Polynomials	Polynomials	Solving Polynomial Equations	Multiplying Polynomials	Multiplying Monomials
		Solving Polynomial Equations	Adding and Subtractions Polynomials	Factoring Special Cases	Dividing Polynomials	Multiplying a Polynomial by a Monomial
		Multiplying Polynomials	Multiplying a Polynomial by a Monomial	Adding and Subtracting Polynomials		Multiplying Two Polynomials
		Dividing Polynomials	Multiplying Polynomials			The Remainder and Factor Theorems
		Rational Zero Theorem	Monomials			
			Polynomials			
			Rational Zero Theorem			
			Dividing Polynomials			
	Complex Numbers and Operations	Complex Numbers	Complex Numbers	Complex Numbers	Operations With Complex Numbers	Polar Coordinates and Complex Numbers
					Complex Numbers and Roots	Real Numbers and Complex Numbers
	Fundamental Theorem of Algebra	Roots and Zeros	Roots and Zeros	The Fundamental Theorem of Algebra	Finding Real Roots of Polynomial Equations	Theory of Polynomial Equations
					Fundamental Theorem of Algebra	
	Binomial Coefficients (and Pascal's Triangle)	The Binomial Theorem	The Binomial Theorem	The Binomial Theorem		Binomial Expansion
	Mathematical Induction and the Binomial Theorem	The Binomial Theorem	The Binomial Theorem	The Binomial Theorem	Mathematical Induction and Infinite Geometric Series	Binomial Expansion
		Proof and Mathematical Induction	Exponential and Binomial Distribution		Binomial Distributions	
			Proof and Mathematical Induction			
Combinatorics and Finite Probability	**Combinatorics and Finite Probability**	Probability of Compound Events	Permutations and Combinations	Probability of Compound Events	Combinations and Permutations	Probability
		Probability Simulations	Probability of Compound Events	Counting Methods and Permutations	Compound Events	Permutations and Combinations
		Probability	Probability: Simple Probability and Odds	Combinations	Permutations and Combinations	Fundamental Counting Principles
		Multiplying Probabilities	Probability Simulations	Probability	Independent and Dependent Events	
		Adding Probabilities	Multiplying Probabilities			
		Counting Outcomes	The Counting Principle	Conditional Probability		
		The Counting Principle	Permutations and Combinations	Permutations and Combinations		
		Permutations and Combinations	Counting Outcomes	Probability of Multiple Events		
			Probability			
			Adding Probabilities			

Note: The Major Topics of School Algebra can be found on page 5. The chapter headings of each textbook reviewed are sorted into each Major Topic of School Algebra category as applicable. If a column only has one box under each Major Topic of School Algebra, it means that that particular textbook had only one chapter or section that was applicable to the specific Major Topic of School Algebra. If a column is empty, that book did not have a chapter or section that fit.

Source: Institute for Defense Analysis Science and Technology Policy.

APPENDIX B: Errors in Algebra Textbooks

The National Mathematics Advisory Panel commissioned a mathematician to look systematically for mathematical errors in

A. two widely used algebra textbooks, one Algebra I and one Algebra II, and
B. a chapter on linear equations in each of three other popular Algebra I textbooks.

A summary of the results is provided below.

(A) **Error density** of an Algebra I and Algebra II textbook is defined to be the following quotient expressed as a percent:

$$\frac{\text{the total number of errors}}{\text{the total number of pages in the book}}$$

It was found that for the review noted above:

- Algebra I book has error density 50.2%, and
- Algebra II book has error density 41%.

This means that, for the Algebra I book, there is on average at least one error every two pages. The Algebra II book is slightly better in this regard, with about four errors in every 10 pages on average.

The analysis also provides additional information regarding the errors found within the Algebra I and Algebra II books. There are three types:

Type I: lack of clarity, minor errors, or misprints.
Type III: a gap in a logical argument or an error on a conceptual level.
Type II: an error that falls between those two types of errors.

The following table summarizes the error densities of these errors in both books:

Table B-1: Error Densities of Errors in Algebra I and Algebra II Textbooks

Book	Type I	Type II	Type III
Algebra I	20.4%	19.5%	10.3%
Algebra II	12.1%	19.4%	9.6%

An example of a Type I error is the statement that all lines with the same slope are parallel; the correct statement should be: *two distinct lines with the same slope are parallel*. Two examples of a Type II error are:

> pointing out that the method of solving a radical equation leads to an **extraneous solution** but without explaining exactly how or why, and stating that two functions are **inverse functions** of each other (e.g., exp and log) without giving their precise domains of definition.

Several examples of Type III errors are provided here; these are more serious errors:

- Graphing a function with a discrete domain of definition (e.g., the price of *n* articles) as a (continuous) straight line;
- Interpreting an event with a probability of 0 as an impossible event, and an event with a probability 1 as one that will definitely occur without specifying that this holds only for a finite sample space;
- Giving the first few terms of a pattern and extending it to the *n*-th terms as if the extension is unique;
- Using technical terms (e.g., **linear regression**) in a problem without giving their definitions;
- Conflating the *definition* of the **negative powers** and **rational powers** of a number with a *theorem*;
- Defining the **slope** of a line using two points on the line without pointing out the independence of the choice of the two points used, and later on;
- Pointing out such an independence without indicating that there is an explanation;
- Proving a general theorem (e.g., a law of exponents) by use of only two or three examples; and
- Giving the *procedure* of the long division of polynomials without explaining what it is about, i.e., never defining **division with a remainder**.

Readers should keep in mind that the error density of Type III errors is about 10% in these two Algebra books, i.e., students are going to find one such error every ten pages on average. This is definitely a cause for concern for both students and teachers.

(B) In this portion of the analysis, one chapter on linear equations in each of three other Algebra I textbooks is analyzed. These three books are referred to as b1, b2, and b3. Because the corresponding chapter in the Algebra I textbook in (A) is also reviewed, this book is referred to as a. Here are the findings of the *error densities* in the chapter on linear equations in these four books:

Table B-2: Error Densities in Chapters on Linear Equations

Book	**Type I**	**Type II**	**Type III**	**Overall**
a	21.7%	16.7%	6.7%	45%
b1	21.2%	6%	6%	33.3%
b2	14.9%	9.2%	3.5%	27.6%
b3	2.9%	2.9%	4.4%	10.3%

Note that two errors of Type III in the book b3 were inadvertently left out by the contractor, but the above computations of error densities did take these overlooked errors into account. The Type III errors involved are the following: One is on not mentioning the fact that the definition of slope of a line is independent of the choice of the two points in the definition, and the other is on not giving an explanation when this independence is mentioned in an example.

This table leaves open the question of whether the book b3, is in fact, significantly better than the rest of the available texts, regarding errors. An independent careful reading of this book suggests that, like the others, there is a concern relative to error frequency. This analysis again raises concern for teachers, students and all others using textbooks. It is imperative that authors, editors and publishers produce mathematically accurate textbooks.