A GENERAL METHOD FOR ESTIMATING THE VARIANCES OF X11-ARIMA ESTIMATORS

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Abstract

The X11-ARIMA method, or X12 method based on the latter procedure, with their various variants are the most commonly procedures used for estimating the seasonally adjusted data and the trend-cycle. Both of these procedures fail to provide estimates for the variances of the estimators that they produce. In this paper we propose a simple general method, based on linear approximation, for estimating the variances of the X11-ARIMA estimators. This method is based on Pfeffermann (1994) but is extended for any multi-stage run. The variances account for the sampling error of the survey estimators and for the variability of the trend, seasonal and irregular components defining the decomposition model. We demonstrate the application of this suggested method to Israeli time series.

Keywords: X11-ARIMA, seasonal adjustment, decomposition model, linear approximation.

Background

Statistical Agencies throughout the world publish each month seasonally adjusted figures for a large number of series. In recent years, some statistical agencies started to publish trend-cycle estimates as a complement to seasonally adjusted data to help reveal better the movements in the series and the occurrence of turning points. The X11-ARIMA method, or X12 method based on the latter procedure, with their various variants are the most commonly procedures used for estimating the seasonally adjusted data and the trend-cycle. However, both of these procedures fail to provide estimates for the variances of the estimators that they produce. Thus, while national statistical institutes publish estimates for the variances of the unadjusted estimators, no estimates are usually published for the variances of the seasonally adjusted estimators and the trend-cycle estimators.

Various approaches for obtaining variances for X11 or X12 estimators have been proposed. Wolter and Monsour (1981) suggested two approaches, one that accounts only for the effect of the sampling error and one that reflects also the uncertainty due to stochastic time series variation. Burridge and Wallis (1985) investigated the use of the steady-state Kalman filter for calculation of model based seasonal adjustment variances for approximating the X11 filters. Pfeffermann (1994) developed an approach that recognizes the contributions of sampling error and irregular variation to the X11 seasonal adjustment variances. Bell and Kramer (1999) developed an approach to obtain variances for X11 adjustments accounting for sampling error and the errors from forecasting extension.

Since May 1996, the Central Bureau of Statistics of Israel uses a specific sequential application of X11-ARIMA method in order to produce seasonally adjusted data and trend estimators: the modified Henderson procedure (see Dagum (1996)). This specific application can be described by the following two steps. In the first step, we apply X11-ARIMA with default options to the original data and estimate the combined linear effect of the trading day and moving Jewish festivals from the irregular estimators. Using these prior adjustment factors, we apply X11-ARIMA in order to obtain the best estimators of the seasonally adjusted data. The second step is carried out through another X11-ARIMA run, with different variants, on the modified adjusted series from step one and produces the final trend estimators. Thus, unlike one-step application of X11-ARIMA, the seasonal component is obtained from step one and the trend-cycle component is obtained from step two. The problem of estimating the variances of the estimators become more difficult in the specific sequential application described above.

One-stage Estimation of the Variances of X11 Estimators by Pfeffermann (1994)

Let { $y_t : t = 1,...,T$ } denote the observed series of the survey estimates. First, we assume the additive decomposition model

$$y_t = T_t + S_t + e_t \tag{1}$$

where T_t is the trend-cycle level, S_t is the seasonal effect and e_t is the compound error term (usually $e_t = I_t + e_t$ where I_t is the irregular component and e_t is the survey error).

It is assumed that $e_t : t = 1,...,T$ are stationary and $E(e_t) = 0$ where the expectation is with respect to all random sources.

Following Young (1968) and Wallis (1982), the seasonal component estimator $\hat{S}_t$ and the trend-cycle estimator $\hat{T}_t$, computed by application of X11 linear filters, can be represented by the linear approximation in the form

$$\hat{T}_t \approx \sum_{k=-(t-1)}^{T-t} w_{kt} y_{t+k} = w_t' y \tag{2}$$

$$\hat{S}_t \approx \sum_{k=-(t-1)}^{T-t} \tilde{w}_{kt} y_{t+k} = \tilde{w}_t' y \tag{3}$$

for t=1,…,T. The matrices W and $\tilde{W}$ can be obtained from respective filters of X11 (programs which calculate these matrices were developed in Statistics Canada and later in BLS).

Under the assumption that the estimators of the seasonal component and the trend level are unbiased we can define the variances that we want to estimate as:

$$\text{var}(\hat{T}_t) \approx \text{var}(\sum_{k=-(t-1)}^{T-t} w_{kt} e_{t+k}), \ \text{t=1,…,T} \tag{4}$$

$$\text{var}(\hat{S}_t) \approx \text{var}(\sum_{k=-(t-1)}^{T-t} \tilde{w}_{kt} e_{t+k}), \ \text{t=1,…,T} \tag{5}$$

$$\text{var}(\hat{T}_t - \hat{T}_{t+s}) \approx \text{var}(\sum_{k=-(t-1)}^{T-t} (w_{kt} - w_{k-s,t+s}) e_{t+k}), \text{t=1,…,T-s} \tag{6}$$

$$\text{var}(\hat{S}_t - \hat{S}_{t+s}) \approx \text{var}(\sum_{k=-(t-1)}^{T-t} (\tilde{w}_{kt} - \tilde{w}_{k-s,t+s}) e_{t+k}), \text{t=1,…,T-s} \tag{7}$$

Since the matrices W and $\tilde{W}$ are known and non-random then the problem of estimating the variances above reduces to estimation of all covariances $\text{cov}(e_t, e_k)$.

Let $\hat{R}_t = y_t - \hat{T}_t - \hat{S}_t$ define the linear filter approximation to the X11 residuals. Then $\hat{R}_t$ can be expressed as

$$\hat{R}_t \approx \sum_{k=-(t-1)}^{T-t} a_{kt} y_{t+k} = \sum_{k=-(t-1)}^{T-t} a_{kt} (T_{t+k} + S_{t+k}) + \sum_{k=-(t-1)}^{T-t} a_{kt} e_{t+k} \tag{8}$$

where $a_{kt} = 1 - w_{kt} - \tilde{w}_{kt}$ if t=k and $a_{kt} = -w_{kt} - \tilde{w}_{kt}$ if $t \neq k$.

Pfeffermann (1994) uses the following two properties in order to estimate the variances of the linear combinations of the error terms e_t from the variances of $\hat{R}_t$.

1. In the center of the series, postulate:

$$\sum_{k=-(t-1)}^{T-t} a_{kt} (T_{t+k} + S_{t+k}) \approx 0 .$$

2. The X11 residual series $\hat{R}_t, t = 25,...,T-24$, is stationary.

Since $\hat{R}_t$ are produced by X11, autocovariances of the series $\hat{R}_t$ can easily be estimated, for example, as

$$\hat{\text{cov}}(\hat{R}_t, \hat{R}_{t-k}) = \frac{1}{T-48}\sum_{t=25+k}^{T-25}(\hat{R}_t - \bar{R})(\hat{R}_{t-k} - \bar{R})$$

and therefore the autocovariances of the series e_t can be obtained by solving the resulting equations for the unknown $\text{cov}(e_t, e_s)$. Note that since $\{e_t, t = 1,...,T\}$ is assumed to be stationary then $\text{cov}(e_t, e_{t+k})$ do not depend on t. In practice, these autocovariances damp to zero so that it can be assumed that, for k>C (cutoff value), $\text{cov}(e_t, e_{t+k}) = 0$. This reduces the number of equations required for the estimation of the variance and the autocovariances of the error terms to C+1 (see Pfeffermann (1994)).
Now, the variances defined by (4)-(7) can be estimated by:

$$\text{var}(\hat{T}_t) \approx \sum_{k=-(t-1)}^{T-t}\sum_{l=-(t-1)}^{T-t} w_{kt}w_{lt}\,\hat{\text{cov}}(e_{t+k}, e_{t+l}), \quad \text{t=1,…,T} \tag{9}$$

$$\text{var}(\hat{S}_t) \approx \sum_{k=-(t-1)}^{T-t}\sum_{l=-(t-1)}^{T-t} \tilde{w}_{kt}\tilde{w}_{lt}\,\hat{\text{cov}}(e_{t+k}, e_{t+l}), \quad \text{t=1,…,T} \tag{10}$$

$$\text{var}(\hat{T}_t - \hat{T}_{t+s}) \approx \sum_{k=-(t-1)}^{T-t}\sum_{l=-(t-1)}^{T-t}(w_{kt} - w_{k-s,t+s})(w_{lt} - w_{l-s,t+s})\,\hat{\text{cov}}(e_{t+k}, e_{t+l}), \quad \text{t=1,…T-s} \tag{11}$$

$$\text{var}(\hat{S}_t - \hat{S}_{t+s}) \approx \sum_{k=-(t-1)}^{T-t}\sum_{l=-(t-1)}^{T-t}(\tilde{w}_{kt} - \tilde{w}_{k-s,t+s})(\tilde{w}_{lt} - \tilde{w}_{l-s,t+s})\,\hat{\text{cov}}(e_{t+k}, e_{t+l}), \quad \text{t=1,…T-s.} \tag{12}$$

Now consider the multiplicative decomposition $y_t = T_t S_t e_t$. It is known that the multiplicative model of X11 gives similar results to the log-additive decomposition, i.e. applying the additive model to the new series $\tilde{y}_t = \log(y_t)$. The method considered above yields $\text{var}(\hat{\tilde{T}})$ and $\text{var}(\hat{\tilde{S}})$. Define $T_t = \exp(\tilde{T}_t)$ and $S_t = \exp(\tilde{S}_t)$. Assume that $\hat{\tilde{T}}_t$ and $\hat{\tilde{S}}_t$ have approximately normal distribution. Utilizing the relationship between the variance of the normal and the lognormal distributions we can write

$$\text{var}(\hat{T}_t) = \hat{T}_t^2\left[\exp(2\,\text{var}(\hat{\tilde{T}}_t)) - \exp(\text{var}(\hat{\tilde{T}}_t))\right] \tag{13}$$

$$\text{var}(\hat{S}_t) = \hat{S}_t^2\left[\exp(2\,\text{var}(\hat{\tilde{S}}_t)) - \exp(\text{var}(\hat{\tilde{S}}_t))\right] \tag{14}$$

(See Pfeffermann (1995) for more details).

Multi-stage Estimation of the Variances of X11-ARIMA Estimators

As described above, the Central Bureau of Statistics of Israel uses a specific sequential application of X11-ARIMA method in order to produce seasonally adjusted data and trend estimators. Thus, the estimates $\hat{S}_t$ and $\hat{T}_t$ are the results of sequential application with different options on each step. Assuming linear approximation is valid on each step, the variances of these estimators can also be estimated by (2) and (3) but with some other matrices W and $\tilde{W}$. These matrices can be obtained analytically if one knows the respective weight matrices on each step but, even in this case, calculations of these matrices can be not trivial. Here, we propose a simple general procedure in order to obtain W and $\tilde{W}$ for any given series $\{y_t\}$.

1. For any given series $\{y_t$, t=1,…,T} and some vector $c = (c_1,...,c_m)$ define T new series $\{y_t^m$, t=1,…,T, m=1,…,T} by $y_t^m = y_t$ if $m \neq t$ and $y_t^m = y_t - c_m$ if $m = t$. Note that for the series $\{y_t^m\}$ all the observations except the m observation coincide with the observations of the original series $\{y_t\}$.
2. Apply all sequence of X11-ARIMA steps independently to all series $\{y_t^m\}$ as defined above in order to obtain the estimates $(\hat{T}_t, \hat{S}_t), (\hat{T}_t^1, \hat{S}_t^1), ..., (\hat{T}_t^T, \hat{S}_t^T)$. Note that the estimates $(\hat{T}_t, \hat{S}_t)$ are obtained from the sequential application of X11-ARIMA to the original series $\{y_t\}$, $(\hat{T}_t^1, \hat{S}_t^1)$ from the same application to series $\{y_t^1\}$,etc.
3. Define the required weights by:

$$w_{kt}^c = \frac{\hat{T}_t - \hat{T}_t^{t+k}}{c_{t+k}}, \qquad \text{k=-(t-1),…,T-t} \tag{15}$$

$$\tilde{w}_{kt}^c = \frac{\hat{S}_t - \hat{S}_t^{t+k}}{c_{t+k}}, \qquad \text{k=-(t-1),…,T-t .} \tag{16}$$

Under the assumption that (2) and (3) hold for the input series $\{y_t\}$, one can easily verify that for any c_m, m=1,…,T the weights w_{kt}^c and $\tilde{w}_{kt}^c$ obtained by this procedure coincide with the required weights.

With real data some or all steps of the estimation usually include some non-linear options such as identification and gradual replacement of outliers, identification and estimation of Arima models for augmenting the series by one year (or two years) of extrapolated values, pre-adjusting for trading day variation, etc. As a result, in general, equations (2) and (3) need not be satisfied with the same weights for any input series. On the other hand even under all the above non-linear options one can expect that for small enough oscillation of input series (2) and (3) are fulfilled with almost the same matrices W and $\tilde{W}$.

Now we can write the properties essential for the estimation of the variances from Pfeffermann (1994) as follows:

$$\hat{T}_t \approx \sum_{k=-(t-1)}^{T-t} w_{kt}^c y_{t+k} \tag{17}$$

$$\hat{S}_t \approx \sum_{k=-(t-1)}^{T-t} \tilde{w}_{kt}^c y_{t+k} \tag{18}$$

$$\hat{R}_t = y_t - \hat{T}_t - \hat{S}_t = \sum_{k=-(t-1)}^{T-t} a_{kt}^c y_{t+k} \approx \sum_{k=-(t-1)}^{T-t} a_{kt}^c e_{t+k} \tag{19}$$

where $a_{kt}^c = 1 - w_{kt}^c - \tilde{w}_{kt}^c$ if $t = k$ and $a_{kt}^c = -w_{kt}^c - \tilde{w}_{kt}^c$ if $t \neq k$. The residual series $\{\hat{R}_t\}$ is stationary in the center of the series.

In the one-stage additive case, the respective weights are known and thus it is very convenient to reduce the problem of variance estimation of multiplicative X11-ARIMA estimators to estimating their log-additive analogue. In our case, since we do not assume that the weights for log-additive analogue of the series to be known, a more simple and direct method can be considered. Following the arguments in the previous section, the multiplicative multi-stage X11-ARIMA estimates $\hat{T}_t$ and $\hat{S}_t$ may be approximated by

$$\log(\hat{T}_t) \approx \sum_{k=-(t-1)}^{T-t} w_{kt}^c \log(y_{t+k}) \quad (20)$$

$$\log(\hat{S}_t) \approx \sum_{k=-(t-1)}^{T-t} \tilde{w}_{kt}^c \log(y_{t+k}). \quad (21)$$

Note that the above procedure for estimating the weights is applied to the logarithms of the original series and the estimates.

Testing the Estimated Matrices of Weights

Let

$$d_t(T) = \hat{T}_t - \sum_{k=-(t-1)}^{T-t} w_{kt}^c y_{t+k}$$

and

$$d_t(S) = \hat{S}_t - \sum_{k=-(t-1)}^{T-t} \tilde{w}_{kt}^c y_{t+k}$$

define the error terms associated with the approximations (17) and (18). Consider

$$s(T) = \left(\frac{1}{T} \sum_{t=1}^{T} d_t^2(T) \right)^{1/2}$$

and

$$s(S) = \left(\frac{1}{T} \sum_{t=1}^{T} d_t^2(S) \right)^{1/2}$$

as measures of exactness of these approximations respectively. These measures can easily be used in order to verify the properties (17) and (18). For example, if these measures are comparably small with respect to the variation in the original series given t, then the weights can be regarded as satisfactory.

Let us now consider (19). Since e_t is not accessible, the exactness of the approximation can not be checked directly. On the other hand, for any sufficiently smooth function g_t (candidate for trend) it is assumed that

$$\sum_{k=-(t-1)}^{T-t} a_{kt}^c g_{t+k} \approx 0.$$

Fit a regression model for the input series y_t, on the data suitable (not very complicated) assuming that the residuals are independent. Let $\tilde{e}_t$ denote to the residuals of the regression. Note that $\tilde{e}_t \neq e_t$ but there exist a sufficiently smooth function $\tilde{g}_t$ so that $\tilde{e}_t = e_t + \tilde{g}_t$. Let

$$d_t(e) = \sum_{k=-(t-1)}^{T-t} a_{kt}^c y_{t+k} - \sum_{k=-(t-1)}^{T-t} a_{kt}^c \tilde{e}_{t+k}$$

define the error term associated with approximation (19). Consider

$$s(e) = \left(\frac{1}{T} \sum_{t=1}^{T} d_t^2(e) \right)^{1/2}$$

as a measure of exactness of the approximation (19).

The above analysis can help to define the vector c and thus the required weights. Note that for any c_m if the statistics suggested above have high values then linear approximation cannot be used for the analysis.

Real Data Examples

This section contains two real data examples that illustrate the application of the method. For each data set the following steps were carried out:

1. Consider sequentially five vectors of constants **c**=(c,…,c) where c=1.1, 1.01, 1.001, 1.0001, 1.00001.
2. For any given series $\{y_t\}$ and each vector **c** define auxiliary series $\{y_t^m\}$, m=1,…,t such that $y_t^m = y_t$ if $m \neq t$ and $y_m^m = y_m / c$.
3. Apply all the sequence of X11-ARIMA steps independently to all series $\{y_t^m\}$ in order to obtain the multi-stage X11-ARIMA estimators $\hat{T}_t$, $\hat{T}_t^m$, $\hat{S}_t$ and $\hat{S}_t^m$ for t=1,…,T and m=1,…,T.
4. For each vector **c** calculate matrices W and $\tilde{W}$ as
$$w_{kt}^c = (\log(\hat{T}_t) - \log(\hat{T}_t^{t+k})) / \log(c_{t+k})$$
and
$$\tilde{w}_{kt}^c = (\log(\hat{S}_t) - \log(\hat{S}_t^{t+k})) / \log(c_{t+k}) .$$
Five pairs of matrices W and $\tilde{W}$ corresponding to five vectors **c** are calculated.
5. Calculate the statistics $d_t(T)$, $d_t(S)$ and $d_t(e)$ as defined above. Use the residuals of 3-d order polynomial regression of $\log(y_t)$ against t, t^2 and t^3 in order to compute $d_t(e)$. If all of the following conditions:
   (One) in the center of the series the weights are time invariant:
   $$w_{i,k}^c \approx w_{i+j,k+j}^c,\ \tilde{w}_{i,k}^c \approx \tilde{w}_{i+j,k+j} \quad \text{for 25<i,k and i+j, k+j<T-25;}$$
   (Two) the s -statistics are comparably small with respect to variation of $\{y_t \setminus t\}$:
   $$\max(s(T), s(S), s(e)) < std(\tilde{e}_t \setminus t)$$

are satisfied for a vector **c** then the corresponding weight matrices W and $\tilde{W}$ are used and the final variances are calculated by the Pfeffermann (1994) method. If the above conditions are satisfied for more than one vector c, then the weights corresponding to the vector with lowest values of the statistics are used.

6. If one of the conditions of the previous step is not satisfied then the steps 1 - 5 are repeated with linear options of X11-ARIMA. Let
$$d(lin)_t(T) = \hat{T}_t - \sum_{k=-(t-1)}^{T-t} w(lin)_{kt}^c y_{t+k},$$
$$d(lin)_t(S) = \hat{S}_t - \sum_{k=-(t-1)}^{T-t} \tilde{w}(lin)_{kt}^c y_{t+k}$$

define the error terms where $\hat{T}_t$ and $\hat{S}_t$ are trend and seasonal component estimates obtained in step 3. Consider the statistics:
$$s(lin)(T) = \left(\frac{1}{T} \sum_{t=1}^{T} d(lin)_t^2(T) \right)^{1/2}$$
and

$$s\,(lin)(S)=\left(\frac{1}{T}\sum_{t=1}^{T}d\,(lin)_t^2\,(S)\right)^{1/2}.$$

If $\max(s\,(lin)(T), s\,(lin)(S)) < std(\tilde{e}_t \setminus t)$ then the weights $W(lin)$ and $\tilde{W}(lin)$ are used and the final variances are calculated by the Pfeffermann (1994) method. If not, then we conclude that for a given series the method described above cannot be used.

Total Industrial Production Index. The data was processed through X11-ARIMA program for the time span from January 1990 to November 1999 (119 observations) using (i) multiplicative decomposition model; (ii) the best options for seasonal adjustment (including adjustments for trading day and moving festivals) and (iii) the modified Henderson trend estimation.

In step 4, for c=0.0001 the following values of the statistics were obtained: $s\,(T)=0.0249$, $s\,(S)=0.0306$, $s\,(e)=0.0005.$ These values are small enough with respect to $std(\tilde{e}_t \setminus t)\approx 0.05$ so that the second condition in 5 is satisfied. This condition is not satisfied for $c \neq 0.0001.$ On the other hand, the first condition concerning the weights being time invariant is not satisfied for the weights when c=0.0001. In step 6, the following values were obtained: $s\,(lin)(T)=0.0264$, $s\,(lin)(S)=0.0351$ and they are almost equal for all vectors **c.** The values of the weights almost coincide in the center of the series with the respective weights obtained by the program developed in BLS. Clearly, the condition concerning the weights being time invariant is satisfied and therefore we can apply the weights $W(lin)$ and $\tilde{W}(lin)$. We should mention that we have compared the X11-ARIMA estimators with the estimators based on W and $\tilde{W}$, and the estimators based on $W(lin)$ and $\tilde{W}(lin)$. These comparisons are not demonstrated here for reasons of space. Figure 1 displays the plot of the trend-cycle estimators bounded by 2 times plus or minus the estimates of the standard deviation of the trend-cycle estimators for the series Total Industrial Production Index. Figure 2 shows the corresponding plot for the seasonal component.

Total Tourist Arrivals by Air. The data was processed through X11-ARAMA program for the time span from January 1980 to December 1994 (168 observations) using (i) multiplicative decomposition model; (ii) the best options for seasonal adjustment (including adjustments for trading day and moving festivals) and (iii) the modified Henderson trend estimation.

In step 5, none of the conditions were satisfied for any vector **c**. In step 6, the following values were obtained: $s\,(lin)(T)=0.0146$, $s\,(lin)(S)=0.0074$ and these values are small enough with respect to $std(\tilde{e}_t \setminus t)\approx 0.015$. Clearly, the condition concerning the weights being time invariant is satisfied and therefore we can apply the weights $W(lin)$ and $\tilde{W}(lin)$. Figure 3 displays the plot of the trend-cycle estimators bounded with 2 times plus or minus the estimates of the standard deviation of the trend-cycle estimators for the series Total Tourist Arrivals by Air for the first 132 points. Figure 4 shows the corresponding plot for the seasonal component.

Concluding Remarks

This paper presented a simple general method, based on Pfeffermann (1994) but extended for any multi-stage run , for estimating the variances of the X11-ARIMA estimators. The variances estimated account for the sampling error and the irregular variation.

Under the assumption that the linear approximation holds, we have presented an alternative procedure for estimating the weight matrices W and $\tilde{W}$. One notable feature of this procedure is that it is very simple and can be applied to any length of series processed through X11-ARIMA program (or X12). On the other hand, the use of very small c-constants may cause, for example, calculation errors, therefore some checking is needed.

As mentioned before, the X11-ARIMA algorithm contains several non-linear options. Even under some non-linear options, one can expect that the linear approximation hold with very similar weight matrices. On the other hand, some non-linear options, for example, the replacement of extreme values, can lead to terrible behavior of the weights but the final estimator can be very close to the respective one obtained through linear operations. We have suggested several statistics in order to determine when the linear weights can be used for the estimation and demonstrated this for real data examples.

Figure1. Total Industrial Production Index
Trend-cycle Estimator

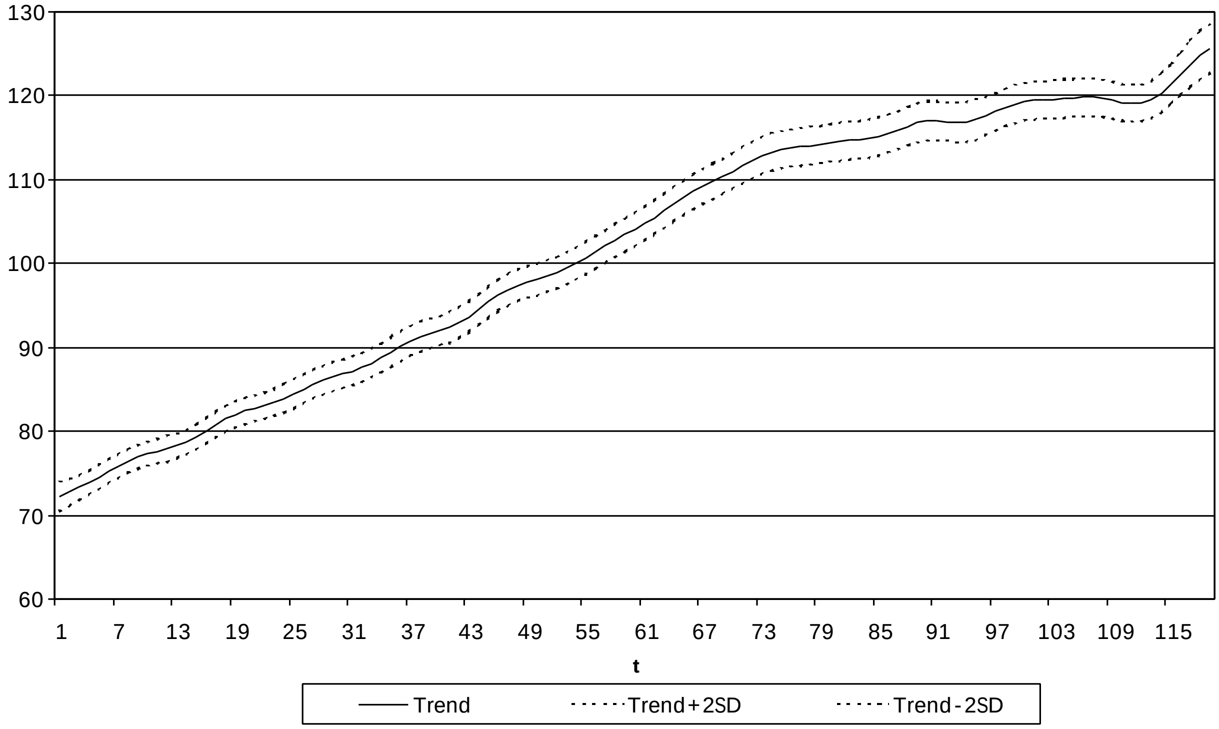


Figure2. Total Industrial Production Index
Seasonal Component Estimator

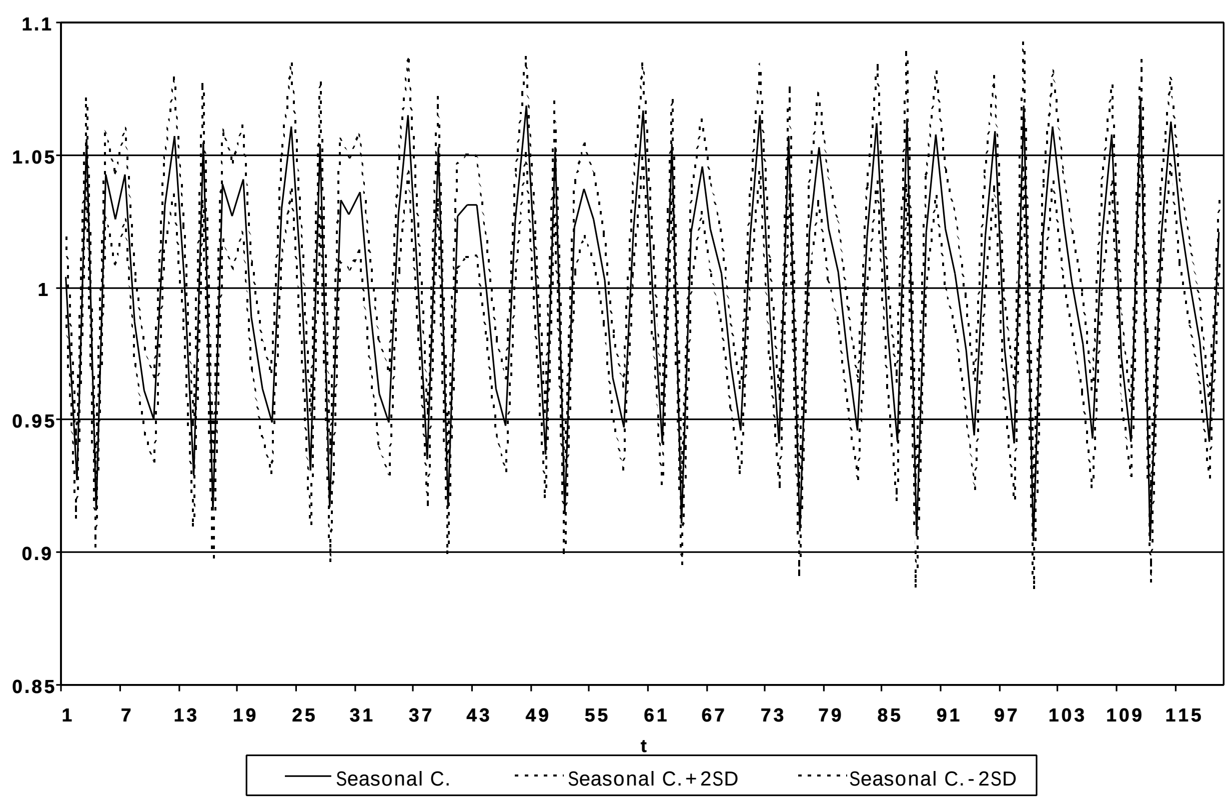


**Figure 3. Total Tourist Arrivals by Air
Trend-cycle Estimator**

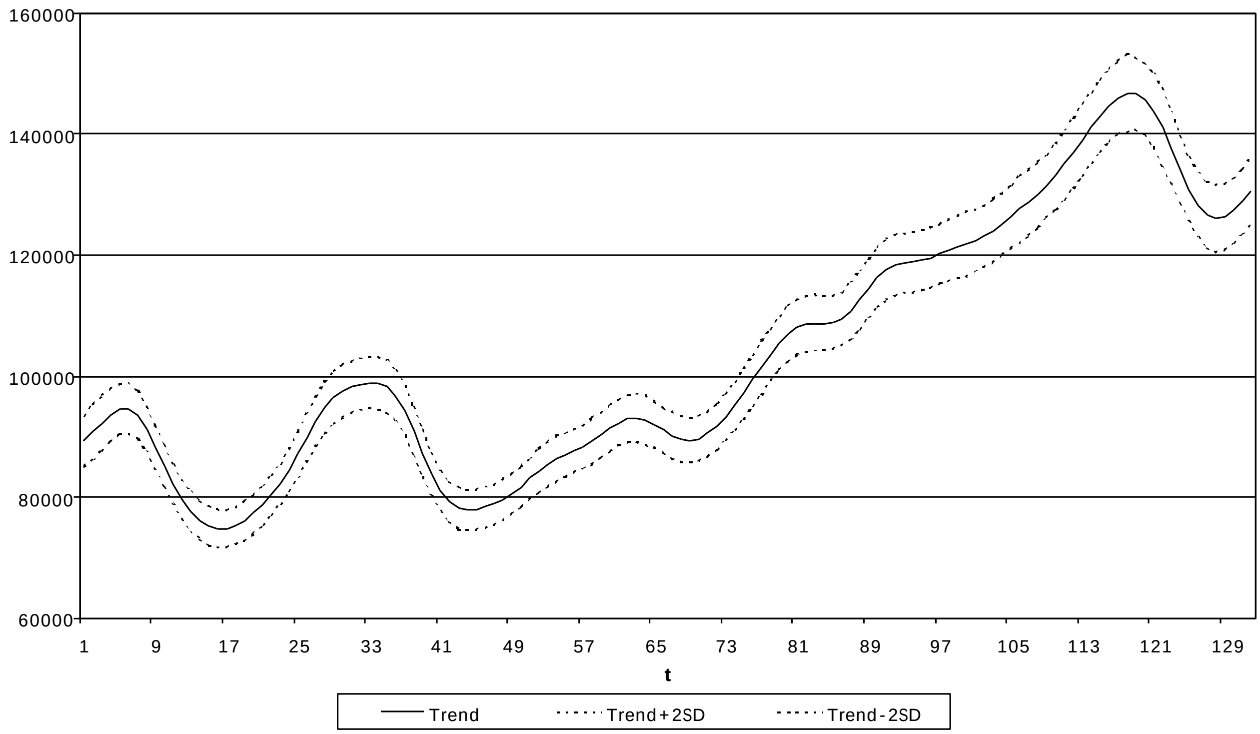
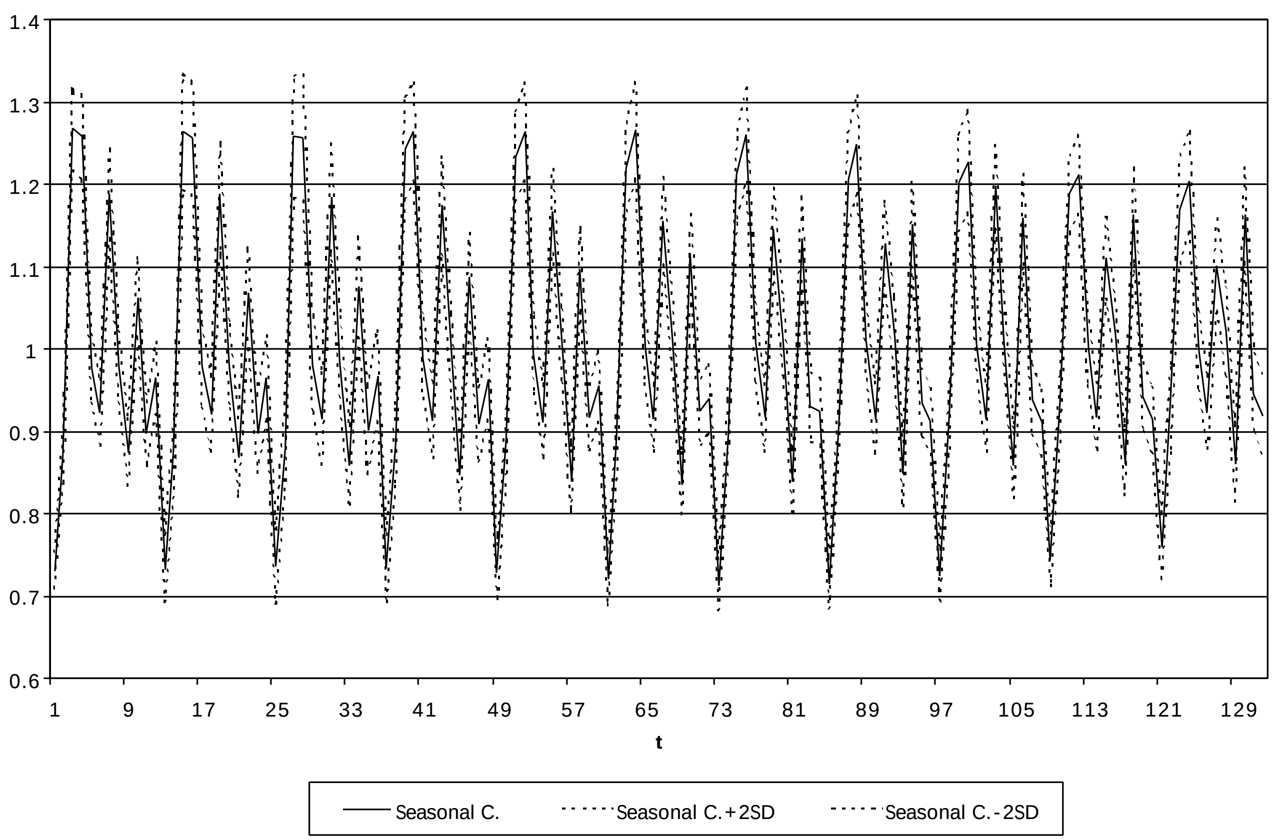


**Figure4. Total Tourist Arrivals by A
Seasonal Component Estimator**



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