$\widehat{\text{THEO}}$1: CHARACTERIZATION OF VERY LONG-TERM FREQUENCY STABILITY*

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Abstract

$\widehat{\text{Theo}}$1 is the first new species of variance that addresses a particularly difficult measurement problem, namely, obtaining reliable estimation of frequency stability for sample periods that are long compared to the length of a data run. $\widehat{\text{Theo}}$1 has statistical properties that are like the Allan variance (Avar), but $\widehat{\text{Theo}}$1 also has two advantages over other estimators of frequency stability: (1) it can evaluate frequency stability at a sample period (τ) of 3/4 the length of a data run, and (2) it presently attains the highest equivalent degrees of freedom (edf) of any estimator of frequency stability including Total-var and overlapping-Avar. $\widehat{\text{Theo}}$1 is unbiased relative to Avar for WHFM noise. $\widehat{\text{Theo}}$1 is biased slightly low with FLFM and RWFM, and we present a formula for a hybrid statistic (TheoH) made up of a combination of $\widehat{\text{Theo}}$1 and Avar in which bias is automatically removed. We explain the sampling function used in $\widehat{\text{Theo}}$1 and show that its frequency response is nearly ideal for extracting power-law noise processes of the types encountered with precision oscillators and clocks. We present results which, for a given data run, show how $\widehat{\text{Theo}}$1 anticipates the levels of frequency stability that are determined by Avar when given a longer data run from the same set of clocks.

1 Introduction

The primary strength of the Allan variance is its half-octave frequency response for a fixed τ_s, where τ_s is a "stride" described in Section 3. Avar's peak response is at reciprocal period of $f_p = \frac{1}{2\tau_s}$. A weakness of the Allan variance is that it cannot characterize frequency stability over an interval τ greater that one-half the length of the data run. For example, suppose we measure the time error between two clocks or oscillators, say, every couple of hours for one month. The maximum-overlap Allan variance estimator of frequency stability cannot report frequency stability for intervals longer than half the month, or two weeks [9, 15, 16]. By definition, a zero-dead-time average frequency difference for averaging interval τ cannot possibly extend beyond 50 % of the length of the data run T, that is, beyond $\tau = \frac{T}{2}$. Furthermore, this estimate is often too low. This is because the chi-square distribution function associated with an estimate composed of only one sample at $\tau = \frac{T}{2}$ (representing one degree of freedom) is so negatively skewed that it is twice as likely to be lower than above the FM noise level's true value [10, 23]. In addition, if a sample estimate of frequency drift is removed, Avar is likely to respond with levels too low at longest-term compared to the expected or true underlying characteristic level [10]. The overlapping estimator for the Allan variance has sufficiently good confidence at short- and medium-term τ averaging intervals but, to be conservative in light of the reasons just stated, it is not recommended for τ beyond 10 % of a data run T [15]. In the one-month example above, this amounts to only a three-day τ-average. In this situation, the best estimator of the Allan variance, which is the Total variance, or Totvar [7, 10, 15], is recommended. Use of the Total approach yields improved confidence between 10 % and 50 % of a data run, or up to two weeks in a one-month data run. At this writing, analysts in our field are confident of Totvar's properties. Easy-to-use 32-bit Windows software is commercially distributed that implements Totvar on large data sets, computes its confidence intervals, and automatically adjusts for bias [22].

It would seem preposterous to report a reliable estimate of frequency stability at a τ of <u>three weeks</u>, given a one-month data run, again considering the reasons stated, not to mention that this is theoretically impossible with the Allan variance! In this paper, we discuss $\widehat{\text{Theo}}$1, a special-purpose statistic that evaluates very-long-term frequency stability at τ between $\frac{T}{2}$ and T, is less susceptible to drift removal, and has a more symmetric distribution function than that of chi-square [6]. At this writing, the statistic has the highest confidence in estimating long-term frequency stability.

2 Sampling Function

Based on the experience gained from Totvar [5, 10, 12] we can manipulate frequency response while maintaining desirable statistical properties. The development of $\widehat{\text{Theo1}}$ involved the following issues: First, it is common practice to measure samples of the time-error function $x(t)$ between two oscillators and then derive frequency stability. For example, Avar is usually calculated as a normalized second difference of time-error measurements $\{x_n\}$. Measuring in this way assures Avar's statistical requirement for zero dead-time between average frequency differences [2]. Second, we desire a frequency response that efficiently extracts levels of common nonstationary FM power-law noise types [1, 14, 18, 19, 21, 24] while retaining simple, distinct straight-line mapping (on log-log plots) to $S_y(f)$, which is the recommended characterization of frequency stability [3]. Finally, we want to maximize equivalent degrees of freedom (edf) for a data run while minimizing bias relative to the conventional Allan variance. We can accomplish these goals by using most, and preferably all, of the available $\{x_n\}$ data, including small sampling interval $\tau_0 \ll T$. Starting with a sequence of time-error samples $\{x_n : n = 1, \ldots, N_x\}$ with a sampling period between adjacent observations given by τ_0, $\widehat{\text{Theo1}}$ averages every permissible squared second-difference of time errors in a given span or stride $\tau_s = 0.75m\tau_0$ as shown in Figure 1. It is defined in terms of $\{x_n\}$ data by

$$\widehat{\text{Theo1}}(m, \tau_0, N_x) = \frac{1}{0.75(N_x - m)(m\tau_0)^2} \sum_{i=1}^{N_x - m} \sum_{\delta=0}^{\frac{m}{2}-1} \frac{1}{(\frac{m}{2} - \delta)} \left[(x_i - x_{i-\delta+\frac{m}{2}}) + (x_{i+m} - x_{i+\delta+\frac{m}{2}})\right]^2, \quad (1)$$

for m even, $10 \leq m \leq N_x - 1$. The sampling functions of $\widehat{\text{Theo1}}$ are easier to understand intuitively in terms of fractional frequency measurements $\{y_n\}$ as shown in Figure 1, where $\{y_n\}$ is defined in terms of $\{x_n\}$ as

$$y_n(\tau) = \frac{x_n - x_{n-1}}{\tau}. \quad (2)$$

Figure 1 shows $\widehat{\text{Theo1}}$'s sampling of fractional-frequency

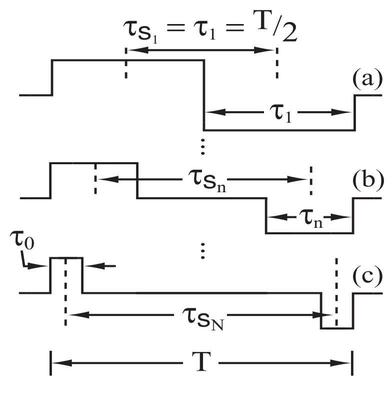


Figure 1: Sampling using $\widehat{\text{Theo1}}$ of fractional-frequency measurements $\{y_n\}$ at varying *stride* values, $\tau_{s1,s2,etc.}$.

measurements $\{y_n\}$. $\widehat{\text{Theo1}}$ computes frequency differences in interval T at varying *stride* $\tau_{s1,s2,etc.}$ and corresponding averaging time $\tau_{1,2,etc.}$ given by the inner summation in Equation (1). The summation's first term ($\delta = 0$) is the sampling in (a) which is that of the classical Allan variance. In this case, stride τ_{s1} equals averaging time τ_1, and both equal $\frac{T}{2}$. For $1 < \delta \leq \frac{m}{2}$, intermediate sampling functions are illustrated by (b) in which $\tau_{s(\cdot)} > \frac{T}{2}$. The summation's last sampling function is (c) in which $\tau_{s(N)} = T - \tau_0$. Therefore, the effective τ-value of the individual frequency differences averaged in $\widehat{\text{Theo1}}$ is between $\frac{T}{2}$ and $T - \tau_0$.

3 Bias

3.1 The Bias Function

Because of the novel data sampling of $\widehat{\text{Theo1}}$ (see Figure 1), there is an inevitable bias with respect to the Allan variance or "Avar". Bias in this instance is the ratio of the expected value of Avar to $\widehat{\text{Theo1}}$. $\widehat{\text{Theo1}}$ is formulated in order to be unbiased with respect to Avar in the case of the white FM (WHFM) noise type. A single bias value was previously reported for each of the five noise types [13]. However, it has since been found that the bias has a slight dependence on τ. The dependence can be described by the function

$$\text{bias}(\tau) = a + \frac{b}{\tau^c}, \quad (3)$$

where a, b, and c are constants. These constants are summarized for each of the five noise types in Table 1. The bias functions were fits to results of Monte Carlo simulations. Thousands of data sets of different lengths (up to 10^5) were used for computer-generated realizations of the five noise types listed in Table 1.

Table 1: Constant values for the bias functions of $\widehat{\text{Theo1}}$ defined in Equation (3).

Noise	a	b	c
WHPM	0.09	0.74	0.40
FLPM	0.14	0.82	0.30
WHFM	1	0	0
FLFM	1.87	-1.05	0.79
RWFM	2.70	-1.53	0.85

3.2 TheoBR and TheoH

A strategy we might follow for using $\widehat{\text{Theo1}}$ as an estimate for Avar at long values of τ is to calculate $\widehat{\text{Theo1}}$ and correct for the bias using the above table. However, this method assumes we know the noise type at a particular value of τ, which leads us to a difficulty at $\tau > \frac{T}{2}$. A useful method for determining noise type is the B_1 function [4]; however, this function is undefined beyond $\frac{T}{2}$ since Avar is undefined. Since B_1 does not exist for the longest τ values of $\widehat{\text{Theo1}}$, we cannot determine the noise type and hence correct for bias. We can estimate long-term noise types only by noting how much pre-whitening is required for a data run [20].

Another simpler strategy for using $\widehat{\text{Theo1}}$ as an estimator of frequency stability is to just remove a computed bias between $\widehat{\text{Theo1}}$ and Avar for a given data run. An unbiased version of $\widehat{\text{Theo1}}$, called TheoBR (for "Theo bias-removed"), can be written

$$\text{TheoBR}(m, \tau_0, N_x) = \left[\frac{1}{n+1} \sum_{i=0}^{n} \frac{\text{Avar}(m = 9 + 3i, \tau_0, N_x)}{\widehat{\text{Theo1}}(m = 12 + 4i, \tau_0, N_x)} \right] \times \widehat{\text{Theo1}}(m, \tau_0, N_x), \quad (4)$$

where $n = \lfloor \frac{0.1 N_x}{3} - 3 \rfloor$ (where $\lfloor \cdot \rfloor$ denotes the floor function). In this equation, $\widehat{\text{Theo1}}$ is defined as in Equation (1), and Avar has its usual definitions as follows:

$$\text{Avar}\,(m, \tau_0, N_x) = \frac{1}{2\,(m\tau_0)^2\,(N_x - 2m)} \times \sum_{n=m+1}^{N_x - m} (x_{n+m} - 2x_n + x_{n-m})^2\,, \quad (5)$$

for $\tau = m\tau_0$.

In order to get the most complete information over the entire data range, we define a hybrid frequency stability estimator called TheoH as a composite of Avar (m, τ_0, N_x) and TheoBR(m, τ_0, N_x), namely,

$$\text{TheoH}(m, \tau_0, N_x) = \begin{cases} \text{Avar}(m, \tau_0, N_x), & \text{for } 1 \le m < \frac{k}{\tau_0}, \\ \text{TheoBR}(m, \tau_0, N_x), & \text{for } \frac{k}{0.75\tau_0} \le m \le N_x - 1, \\ & m \text{ even}, \end{cases} \quad (6)$$

where k is the largest $\tau \le 10\%T$ where Avar (m, τ_0, N_x) has sufficient confidence. From Equation (4), TheoBR(m, k, N_x) = Avar(m, k, N_x); thus TheoH can be plotted vs. τ as one function with no discontinuity using (6); however, it must be noted that within the definition, Avar and TheoBR have different dependence on τ of $\tau = m\tau_0$ and $\tau = 0.75m\tau_0$ respectively. We note that deviation or square root of Equations (4)-(6) will be reported.

4 Properties of $\widehat{\text{Theo1}}$

4.1 Criteria for $\tau_s = 0.75m\tau_0$

Response of a statistic is the Fourier transform of its sampling sequence that, in some cases, can be nearly impossible to interpret in the time domain but easier to understand in the frequency domain [11]. Recall that it is desirable to maintain Avar's half-octave frequency response with peak at a reciprocal period of $f_p = \frac{1}{2\tau}$. The dashed line in Figure 2 shows the response of a constant-Q, half-octave pass-band filter considered to be ideal for extracting typical power-law noise levels [1, 14, 18, 19, 21, 24].

Frequency-response functions associated with $\widehat{\text{Theo1}}$ and Avar are shown in Figure 2. Prior to Equation (1), we obtained a high-edf, low-bias prototype variance, whose frequency response peak was shifted above $f_p = \frac{1}{2\tau_s}$. We found that if $\tau_s = 0.75m\tau_0$ and the amplitude of the response is adjusted by 0.75 (in the denominator of the amplitude coefficient of definition in Equation (1)), then the frequency response could be shifted to be precisely $f_p = \frac{1}{2\tau_s}$.

4.2 Response to Data Periodicity

Avar has deep nulls in its response to periodic or cyclical variations in $\{x_n\}$ at frequencies $f = \frac{\text{int}}{\tau}$, int = 1, 2, 3, ... , whereas $\widehat{\text{Theo1}}$ does not (see Figure 2). This means that the response of $\widehat{\text{Theo1}}$ to a periodic term in the data with frequency near $f = \frac{\text{int}}{\tau}$ is going to be more accurate than if Avar is used. In the end, $\widehat{\text{Theo1}}$'s frequency response is closer to the response of the ideal pass-band filter that Avar attempts to approximate. This closer approximation explains why $\widehat{\text{Theo1}}$ is so efficient in extracting power-law noise levels and types.

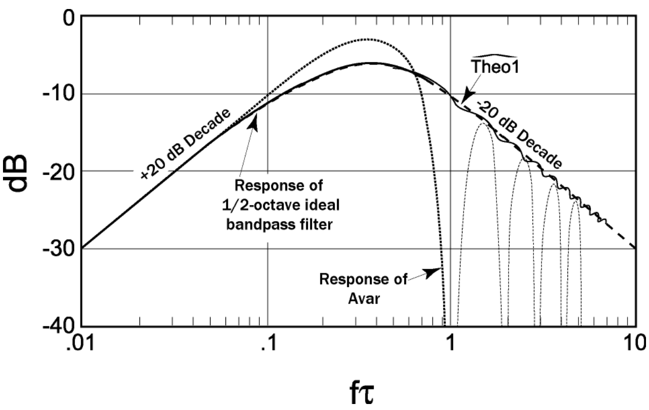


Figure 2: A comparison of frequency responses of $\widehat{\text{Theo1}}$, Avar, and a passband variance consisting of a simple cascade of a single-pole high-pass followed by a low-pass filter with identical break points at $RC = \tau/2$ [24].

5 Extraction of Noises and Other Errors

If the plot of frequency stability is not a straight line over a particular range (for example, it has nulls, "structure" or oscillations), then it is probably not a power-law process and we must try to estimate the spectral density $S_y(f)$ using another method such as a DFT or FFT in addition to using the $\widehat{\text{Theo1}}$-deviation plot. In general, any number of random noise components may be present in the data, depending on the type of test and reference oscillators being compared and the environment in which the data are obtained.

After using a method to extract the level of contribution for each component, relationships of $\widehat{\text{Theo1}}$-deviation slopes, various noise sources, and the corresponding noise level as previously discussed above are summarized in Table 2. Assuming that the computed $\widehat{\text{Theo1}}$-deviation is expressed in *fractional frequency fluctuation* and that τ is in seconds, then the expressions can be used for 1σ measures of level.

Table 2: Mapping of $\widehat{\text{Theo1}}$-deviation level and slope on log-log plot to the square root of noise spectrum, $\sqrt{S_y(f)}$, or rms fractional frequency fluctuations, $\frac{\Delta\nu}{\nu_0}$ of an oscillator pair, in a 1 Hz BW evaluated at Fourier-frequency f. QPM is quantized phase modulation. QSFM(f_0) means quasi-sinusoidal frequency modulation at f_0.

Noise Type	$\widehat{\text{Theo1}}$ Level a_μ	$\widehat{\text{Theo1}}$ Slope $\frac{\mu}{2}$	$\sqrt{S_y(f)}$ or $\left(\frac{\Delta\nu}{\nu_0}\right)_{rms}$ vs. f
QPM*	a_{-2}	-1	$\frac{1}{2\sqrt{3f_h}}\tau a_{-2}$
WH/FL PM*	a_{-2}	-1	$\frac{\pi}{\sqrt{3f_h}}\tau a_{-2}$
WH FM	a_{-1}	$-\frac{1}{2}$	$\sqrt{2\tau}a_{-1}$
FL FM	a_0	0	$\frac{1}{\sqrt{ln2}}a_0$
RW FM	a_{+1}	$+\frac{1}{2}$	$\frac{1}{\pi}\sqrt{\frac{6}{\tau}}a_{+1}$
Drift	a_{+2}	$+1$	$\frac{2}{\tau}a_{+2}$
QSFM(f_0)	$\frac{\Delta\nu}{\nu_0}\left(\frac{sin^2\pi f_o\tau}{\pi f_o\tau}\right)$	$-1(avg)$	$2\frac{\Delta\nu}{\nu_0}(f_o)$

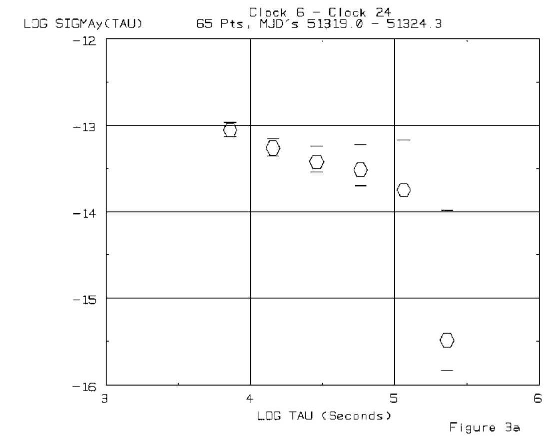
*requires f_h (high-freq. cutoff), $2\pi f_h\tau \gg 1$.

6 Examples

It is well known that interpreting the longest-term frequency stability from a $\sigma_y(\tau)$ plot can be problematic, even misleading, as frequency fluctuations become increasingly nonstationary [8, 17]. It is desirable to see how readily we can identify integer power-law noise types and levels at long-term extremes using log-log plots of $\widehat{\text{Theo1}}$-deviation on actual data as compared with the same data processed with the best (max-overlap) sample Allan deviation $\sigma_y(\tau)$. Accurate noise typing at long term remains essential for at least the following reasons:

1. predicting the evolution of time-error of a single clock,
2. calculating confidence intervals,
3. distinguishing and estimating frequency drift,
4. properly correcting for measurement-system dead-time, if any,
5. optimally combining clocks or frequency standards in an ensemble to form a time scale.

$\widehat{\text{Theo1}}(m, \tau_0, N_x)$ formulated in Equation (1) has been coded into NIST's time-scale computer for testing purposes. Various time-difference $\{x_n\}$ series of clock and time scale comparisons are readily available, data which have been recorded over many years at NIST, and a representative set of example plots are given that illustrate how $\widehat{\text{Theo1}}$-deviation characterizes frequency stability for averaging times up to the full length of a data run. 90 % confidence limits are set according to the straight-line slope projected off the preceeding adjacent power-of-2 τ-value. For slope $\mu \geq -0.5$, FLFM noise is assumed, otherwise WHFM is assumed. Finally, the confidence on the last point of $\widehat{\text{Theo1}}$-deviation plots assumes RWFM if a straight-line does not fit within the confidence limits of the preceeding two shorter-term power-of-2 τ-values, for example, if the last three points oscillate beyond a straight line that is able to fit inside their 90 % confidence limits.



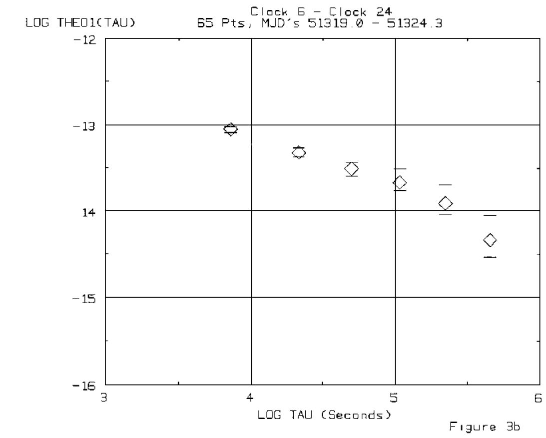


Figure 3: Two high-performance commercial Cs frequency standards in NIST's time scale taken over five days. The bottom plot, which estimates Theo1(τ)-deviation, can characterize fractional-frequency noise level and power-law noise type out to 3/4 of the length of the data run. $\widehat{\text{Theo1}}(\tau)$-deviation also has lower overall uncertainty compared to $\hat{\sigma}_y(\tau)$. Even though the last point drops, WHFM noise type is still supported within $\widehat{\text{Theo1}}(\tau)$'s uncertainties.

In addition to knowing the nature of the oscillators or time scales involved in a particular comparison, the narrower confidence limits of $\widehat{\text{Theo1}}$-deviation are especially useful for singling out a likely underlying candidate power-law noise type. Table 2 allows us to relate the $\widehat{\text{Theo1}}$-deviation amplitude to the amplitude of the spectral density of frequency noise modulation, namely $\sqrt{S_y(f)}$, arguably the most useful function involving frequency-standard noise modeling, oscillator synchronization or clock ensembling for example. Using $\widehat{\text{Theo1}}$-deviation, we are able to confidently estimate spectral-density noise level and type at unprecedented low values of Fourier frequencies.

The first examples (figures 3 and 4) compare a $\sigma_y(\tau)$ plot and a $\widehat{\text{Theo1}}$-deviation plot obtained from two commercial Cs standards for data runs of 5 days and 170 days. In both examples, the characteristic noise type is WHFM out to 3/4 of the 5-day and 170-day lengths of the data sets according to the $\widehat{\text{Theo1}}$-deviation plot.

Figure 5 shows the evolution of frequency stability plots of two high-grade commercial Cs standards in the NIST time scale starting at the same time origin and advancing from one day to over two years of data. These are particularly interesting because the plots show $\widehat{\text{Theo1}}$-deviation values beyond the last $\sigma_y(\tau)$ values and give some feel for whether $\widehat{\text{Theo1}}$-deviation can anticipate longer-term $\sigma_y(\tau)$ values. To begin, figure 5-c shows that $\widehat{\text{Theo1}}$-deviation indicates the onset of non-WHFM (nonstationary) long-term noise at $\tau \approx 85 - 150$, days whereas the $\sigma_y(\tau)$ plot does not to its last calculable value of $\tau = 85$ days. As the data run gets longer, figure 5-d shows that the $\sigma_y(\tau)$ indicates that flicker FM is occuring at this time. This would be reasonable as shown over the last four octaves of τ, out to its

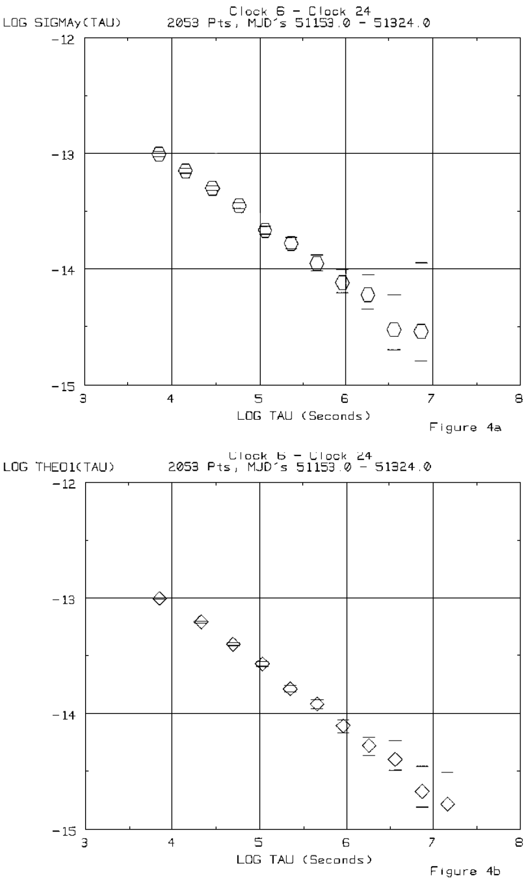


Figure 4: Two high-performance commercial Cs frequency standards from NIST's time scale with data taken over 170 days.

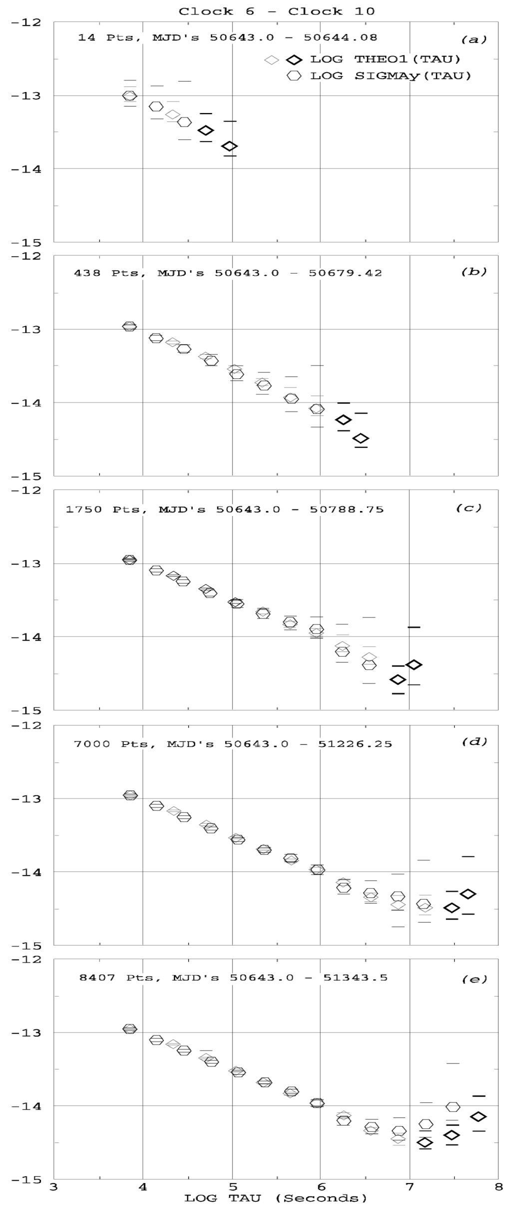


Figure 5: Frequency stability plots that start at the same time representing two high-performance commercial Cs standards in the NIST time scale for (a) one day, (b) 37 days, (c) 146 days, (d) 583 days, and (e) 701 days. The Allan deviation function is identified by the hexagons, and Theo1(τ)-deviation function is identified by the diamonds.

last $\sigma_y(\tau)$-value, corresponding to about 170 days, except that $\widehat{\text{Theo1}}$-deviation contradicts this hypothesis. In fact at the same time, $\widehat{\text{Theo1}}$-deviation supports a hypothesis of WHFM becoming RWFM, or possibly frequency drift, at $\tau \approx 170 - 450$ days since the slope is slightly steeper than RWFM $\propto \tau^{+\frac{1}{2}}$ but not quite as steep as Dr $\propto \tau^{+1}$. Full results using the remaining 120 days are shown in figure 5-e and indicate that $\widehat{\text{Theo1}}$-deviation continues to support long-term RWFM, and now, $\sigma_y(\tau)$ does as well.

7 Conclusion

$\widehat{\text{Theo1}}$ is effective to large τ-values, including 3/4 of the entire data run. This means that longest-term frequency stability can be obtained with only one-third more data-collection time. $\widehat{\text{Theo1}}$, like Avar and Totvar, is invariant to an overall shift in phase and frequency. $\widehat{\text{Theo1}}$, like Avar and Totvar, retains simple straight-line mapping (on log-log plots) to $S_y(f)$ for easily extracting the levels of the usual five FM power-law noise types by a linear-least-squares fit.

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