RESEARCH REPORT SERIES
(Statistics #2006-9)

Exact Formulas for the Hodrick-Prescott Filter

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Report Issued: September 7, 2006

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Abstract

The Hodrick-Prescott (HP) filter is widely used in the field of economics to estimate trends and cycles from time series data. For certain applications – such as deriving implied trend and cycle models and obtaining filter weights – it is desirable to express the frequency response of the HP as the spectral density of an $ARMA$ model; in other words, to accomplish the spectral factorization of the HP filter. This paper presents an exact approach to this problem, which makes it possible to provide exact algebraic formulas for the HP filter coefficients in terms of the HP's signal-noise ratio.

Keywords. Nonstationary time series, filtering, business cycle.

1 Introduction

The Hodrick-Prescott (HP) filter is widely used in the field of economics to estimate trends and cycles from time series data. Although the filter has been informally used in many fields for many decades, it was more recently introduced to the study of business cycles by Hodrick and Prescott (1997). This filter can be interpreted as a Mean Squared Error (MSE) optimal signal extraction filter for the trend in the smooth trend model (see Harvey, 1989); this is also discussed in Harvey and Trimbur (2003). In recent years the filter has been used in the seasonal adjustment program $SEATS$ (Gómez and Maravall, 1997) to produce cycle estimates from the program's trend-cycle output (see Kaiser and Maravall, 2005). Conceptually, this results in "implied" models for the cycle and the trend, such that applying the HP filter results in MSE optimal estimates. This latter paper provides the main application for the present work.

In order to determine these implied models, it is necessary to obtain the spectral factorization of the HP filter. Namely, if we view the frequency response of the HP filter as the spectral density of an $ARMA$ process, the problem of spectral factorization amounts to the determination of the coefficients of the AR polynomial (the MA coefficients are trivial). A knowledge of these AR coefficients, along with the corresponding roots, facilitates the construction of implied cycle and trend models. These models are actually necessary for any finite-sample implementation of the HP filter; if one wishes to use the State Space smoother (Durbin and Koopman, 2001) to obtain cycle estimates, then one must know the actual implied cycle model. Equivalently, a matrix based approach to signal extraction requires knowledge of the autocovariance function of the (differenced) trend and cycle (McElroy, 2005). This application provides the primary motivation for obtaining the spectral factorization of the HP; a secondary motivation lies in determining the exact filter weights of the HP filter. These would typically be computed from a knowledge of the zeroes of the AR polynomial, thus avoiding the need to perform numerical integration.

Of course, the general problem of spectral factorization has been solved, and their are various numerical recipes for computing the polynomial coefficients – see Pollock (1999) for a discussion. These algorithms are numerical, and hence inexact. Note that Kaiser and Maravall (2005) provides an algorithm for computing the exact AR coefficients, though the dependence on the signal-noise ratio (SNR) q is somewhat obscured. The paper at hand provides exact algebraic formulas for the AR coefficients, the zeroes of the AR polynomial, and the HP filter coefficients, all of which only depend upon the SNR q. The advantages of the exact formulas are the following: higher precision of the filter quantities (discussed in Section 4), easier software implementation, faster computation time, and mathematical insight. This last aspect pertains to the HP filter coefficients: the exact formulas allow us to see both the geometric decay of the weights as well as the side-lobe behavior, and how these qualitative features depend on the choice of q. In addition, a simple relationship between q and the cycle-periodicity of an implied cycle model can easily be traced.

The plan of this paper is as follows. Section 2 defines the mathematical problem and traces the derivation of the spectral factorization. We also determine the roots of the AR polynomial associated with the HP, focusing on the factorization that results in zeroes outside the unit circle of complex plane. These results are summarized in equations (10) through (13). In Section 3 we discuss two applications: obtaining the implied cycle and trend models discussed in Kaiser and Maravall (2005) that result from applying the HP filter to a trend-cycle process with known $ARIMA$ structure, and determining exact formulas for the HP filter weights using the theory of residues from complex analysis. The formula for the filter weights is given in (19). We demonstrate the practicality of this work in Section 4 through a simple illustration of these two applications; the formulas are trivial to encode in software. Note that we do not advocate taking an exact approach to the general problem of spectral factorization; it works out well for the HP filter because its associated AR polynomial is of order 2, which is sufficiently low to guarantee success. But since the HP enjoys such a wide use in economics (and other sciences as well), the effort involved in obtaining the exact formulas seems worthwhile.

2 Spectral Factorization

Let B denote the backshift operator, and let $\overline{B} = B^{-1} = F$, so that the "conjugate" of the backshift operator is the forward shift operator F. Now for a given SNR $q > 0$, the HP filter is defined as

$$H(B) = \frac{q}{q + |1 - B|^4}, \qquad (1)$$

which has symmetric coefficients φ_j. Some authors refer to (1) as giving the HP "low-pass" filter, since it is appropriate for trend estimation. The HP "high-pass" filter is just $1 - H(B)$, and is frequently used to estimate cycles from trend-cycle data. For a model-based interpretation of the HP filter, see Harvey and Trimbur (2003). We will focus on the spectral factorization of the HP low-pass (1), since the factorization of the HP high-pass will follow easily. Now although we make reference to spectral methods, our calculations will actually take place in the time domain. The main idea is that the frequency response of the HP filter (1) can be conceived of as an $ARMA$ model, in which case there exist polynomials $\theta(B)$ and $\phi(B)$ and a constant c such that

$$H(B) = \frac{\theta(B)\theta(F)}{\phi(B)\phi(F)}c.$$

Here we follow the Box and Jenkins (1976) convention that the leading coefficient of polynomials should equal 1 (so $\theta(0) = 1 = \phi(0)$), but we do *not* adopt the minus convention for AR polynomials. In particular, $\phi(B) = 1 + \phi_1 B + \phi_2 B^2$; it is simple to see from (1) that we must have $\theta(B) = 1$. So if we view the HP frequency response as the spectral density of an $ARMA$ process, then that process is actually an $AR(2)$. Our task then, is to determine ϕ_1, ϕ_2, and c in terms of q. Mathematically, it will be simpler to formulate the problem equivalently as

$$q + |1 - B|^4 = \psi(B)\psi(F) \qquad (2)$$

where $\psi(B) = x + yB + zB^2$, for x, y, and z to be determined. Once we've solved for these quantities, we can divide through by x to determine ϕ_1, ϕ_2, and c. Now expanding both sides of (2) and matching coefficients of B and F yields the following three equations:

$$1 = x\,z \qquad (3)$$

$$-4 = x\,y + y\,z \qquad (4)$$

$$6 + q = x^2 + y^2 + z^2. \qquad (5)$$

Now these can be combined (or alternatively let $B = 1$ in (2)) to yield

$$q = (x + y + z)^2,$$

from which the algebraic necessity of $q \geq 0$ is clear. From this we obtain $x+y+z = \pm\sqrt{q}$, and we proceed in two cases. First suppose that $x+y+z=\sqrt{q}$. Then

$$x+z=\sqrt{q}-y$$

and by (4), $y^2-\sqrt{q}y-4=0$. Applying the quadratic formula yields

$$y=\frac{\sqrt{q}\pm\sqrt{q+16}}{2}. \tag{6}$$

Hence we have

$$x+z=\frac{\sqrt{q}\mp\sqrt{q+16}}{2}. \tag{7}$$

Multiplying this through by x and using (3), we get

$$x^2+\left(\frac{-\sqrt{q}\pm\sqrt{q+16}}{2}\right)x+1=0.$$

We again apply the quadratic formula, noting that complex roots are a possibility:

$$x=\frac{\sqrt{q}\mp\sqrt{q+16}\pm\sqrt{2q\mp 2\sqrt{q}\sqrt{q+16}}}{4}. \tag{8}$$

This actually refers to a set of four possible solutions; the first $\mp$ and last $\mp$ above must have the same sign, being the opposite of the $\pm$ sign in (6). The $\pm$ in (8) is independent of the $\mp$ signs. Now from (7) we obtain

$$z=\frac{\sqrt{q}\mp\sqrt{q+16}\mp\sqrt{2q\mp 2\sqrt{q}\sqrt{q+16}}}{4}. \tag{9}$$

Here the first and last $\mp$'s are as in (8) above, but the middle $\mp$ must have the opposite sign as the $\pm$ in (8). Now we note that $q-\sqrt{q}\sqrt{q+16}<0$, so that the case $y=(\sqrt{q}+\sqrt{q+16})/2$ results in complex coefficients; we seek real coefficients, so these solutions are rejected. This leaves us the two triples (x,y,z) given by

$$\left(\frac{\sqrt{q}+\sqrt{q+16}+\sqrt{2q+2\sqrt{q}\sqrt{q+16}}}{4}, \frac{\sqrt{q}-\sqrt{q+16}}{2}, \frac{\sqrt{q}+\sqrt{q+16}-\sqrt{2q+2\sqrt{q}\sqrt{q+16}}}{4}\right)$$
$$\left(\frac{\sqrt{q}+\sqrt{q+16}-\sqrt{2q+2\sqrt{q}\sqrt{q+16}}}{4}, \frac{\sqrt{q}-\sqrt{q+16}}{2}, \frac{\sqrt{q}+\sqrt{q+16}+\sqrt{2q+2\sqrt{q}\sqrt{q+16}}}{4}\right).$$

Now the same type of analysis in the case that $x+y+z=-\sqrt{q}$ provides another two solutions:

$$\left(\frac{-\sqrt{q}-\sqrt{q+16}+\sqrt{2q+2\sqrt{q}\sqrt{q+16}}}{4}, \frac{-\sqrt{q}+\sqrt{q+16}}{2}, \frac{-\sqrt{q}-\sqrt{q+16}-\sqrt{2q+2\sqrt{q}\sqrt{q+16}}}{4}\right)$$
$$\left(\frac{-\sqrt{q}-\sqrt{q+16}-\sqrt{2q+2\sqrt{q}\sqrt{q+16}}}{4}, \frac{-\sqrt{q}+\sqrt{q+16}}{2}, \frac{-\sqrt{q}-\sqrt{q+16}+\sqrt{2q+2\sqrt{q}\sqrt{q+16}}}{4}\right).$$

However, these latter two triples are just minus one times the first two triples. At this point, we divide off by x (note by (3) that it must be nonzero) and obtain the two solution pairs (ϕ_1, ϕ_2):

$$\left(\frac{2(\sqrt{q} - \sqrt{q+16})}{\sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}}, \frac{\sqrt{q} + \sqrt{q+16} - \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}}{\sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}} \right)$$

$$\left(\frac{2(\sqrt{q} - \sqrt{q+16})}{\sqrt{q} + \sqrt{q+16} - \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}}, \frac{\sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}}{\sqrt{q} + \sqrt{q+16} - \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}} \right).$$

Of course, this recognizes that x has two possible signs; but the sign of x will be irrelevant in the AR representation (see below). In particular, we note that $c = q/x^2$. Now one of the above solutions corresponds to $\phi(B)$ with roots outside the unit circle, and the other with roots inside; we prefer the representation indicated by the former solution, since it will generate a stationary AR representation. The discriminant for the former solution pair is

$$\phi_1^2 - 4\phi_2 = \frac{8(q - \sqrt{q}\sqrt{q+16})}{\left(\sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}\right)^2},$$

which is negative, indicating complex roots. The roots ζ are

$$\zeta = \frac{-(\sqrt{q} - \sqrt{q+16}) \pm i\sqrt{2\sqrt{q}\sqrt{q+16} - 2q}}{\sqrt{q} + \sqrt{q+16} - \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}},$$

where $i = \sqrt{-1}$. The squared magnitude works out to

$$|\zeta|^2 = \frac{16}{\left(\sqrt{q} + \sqrt{q+16} - \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}\right)^2}.$$

Now $\sqrt{q} + \sqrt{q+16} - \sqrt{2q + 2\sqrt{q}\sqrt{q+16}} < 4$ iff

$$(\sqrt{q} + \sqrt{q+16})^2 - \left(\sqrt{2q + 2\sqrt{q}\sqrt{q+16}}\right)^2 < 4(\sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}),$$

which holds iff

$$4 < \sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}.$$

But this latter condition is always true for $q > 0$, which implies that $|\zeta|^2 > 1$, indicating that the former pair of solutions have roots outside the unit circle. In summary, the invertible (stationary)

AR solution is given by

$$c = \frac{16q}{\left(\sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}\right)^2} \tag{10}$$

$$\phi_1 = \frac{2(\sqrt{q} - \sqrt{q+16})}{\sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}} \tag{11}$$

$$\phi_2 = \frac{\sqrt{q} + \sqrt{q+16} - \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}}{\sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}} \tag{12}$$

$$\zeta = \frac{(\sqrt{q+16} - \sqrt{q}) \pm i\sqrt{2\sqrt{q}\sqrt{q+16} - 2q}}{\sqrt{q} + \sqrt{q+16} - \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}}. \tag{13}$$

Of course an explosive AR solution also exists, but there seems to be little interest in this given the intended applications. The behavior of the roots can be further investigated by writing $\zeta = |\zeta|e^{i\theta}$, now letting ζ denote the root in the first quadrant. We at once obtain

$$|\zeta| = \left(\sqrt{q} + \sqrt{q+16} + \sqrt{2q + 2\sqrt{q}\sqrt{q+16}}\right)/4 \tag{14}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{2\sqrt{q}\sqrt{q+16} - 2q}}{\sqrt{q+16} - \sqrt{q}}\right) = \tan^{-1}\left(\sqrt{2q + 2\sqrt{q}\sqrt{q+16}}/4\right). \tag{15}$$

Now (14) tells us that $|\zeta|$ is increasing in q at the rate of $q^{1/2}$. Also $\pi/2 - \tan^{-1}(\sqrt{q}) \sim 1/\sqrt{q}$ as $q \to \infty$, which gives a sense of θ's dependence on q asymptotically.

3 Applications

Our first application stems from the work of Kaiser and Maravall (2005), which forms a primary motivation for this paper. Suppose that we are given a process Y_t with a known $ARIMA$ structure. For example, following Kaiser and Maravall (2005) we might think of Y_t as the trend-cycle estimate that is the output of some model-based signal extraction procedure; assuming that the model used for the signal and noise components match the Data Generation Process (DGP), it would be possible in theory to know the exact $ARIMA$ model of Y_t. Or Y_t might just be some data, for which we believe that our $ARIMA$ model matches the DGP. In any event, we describe Y_t via the difference equation

$$\varphi(B)Y_t = \vartheta(B)\xi_t, \tag{16}$$

where ξ_t is white noise, denoted by $WN(0, \sigma^2)$. The polynomials φ and ϑ follow the same conventions discussed in Section 2, but note that they may contain unit roots. Now if we apply the HP filter to Y_t, the resulting output has pseudo-spectral density

$$H(e^{-i\lambda})f_Y(\lambda) = \frac{q}{q + |1 - e^{-i\lambda}|^4} \frac{|\vartheta(e^{-i\lambda})|^2}{|\varphi(e^{-i\lambda})|^2}\sigma^2.$$

The concept in Kaiser and Maravall (2005) is to let this be the pseudo-spectral density for the trend component T_t; likewise, $(1 - H(e^{-i\lambda}))f_Y(\lambda)$ is assigned to be the spectral density of the cycle C_t. Note that we typically want cycles to follow stationary models, so the $|1 - e^{-i\lambda}|^4$ factor in the numerator of $(1 - H(e^{-i\lambda}))$ should cancel with any unit root factors in the denominator of the pseudo-spectrum $f_Y(\lambda)$. Using our spectral factorization results from Section 2, we find that the trend and cycle have the following implied models:

$$\phi(B)\varphi(B)C_t = (1 - B)^2\vartheta(B)\xi_t^C \tag{17}$$

$$\phi(B)\varphi(B)T_t = \vartheta(B)\xi_t^T, \tag{18}$$

where ξ_t^C is $WN(0, \sigma^2 c/q)$ and ξ_t^T is $WN(0, \sigma^2 c)$. Again, the unit root factors of φ (if any) must be canceled by the $(1 - B)^2$ factor in the cycle model in order for the cycle to be stationary. The implied models (17) and (18) are easily computed from (10)-(13) and a knowledge of the parameters in (16). Now as discussed in Kaiser and Maravall (2005), the leading frequency associated with a stochastic cycle following an $ARMA$ model of the type (17) is given by the angular portion of the AR roots; in this case the frequency is θ (15). The formula $2\pi/\theta$ gives the period corresponding to frequency θ, and this should be renormalized to determine the period in years; if the data is monthly, we divide by 12. Clearly, (15) allows us to determine the exact functional relationship between q and the implied cycle period; this is illustrated in Figure 1 below. Values of q were chosen at intervals of 10^{-4}, ranging from .2 down to zero. For this range, the cycle period varies between roughly one and seven years. This plot and the accompanying formula provide cycle analysts with a simple method of determining q, given a desired cycle periodicity.

Our second application involves the computation of exact HP filter weights; here we consider the low-pass weights associated with (1). The jth weight φ_j is the integral of the frequency response against $e^{ij\lambda}$; since the frequency response is an even function, it suffices to consider $j \geq 0$, as the weights will be symmetric. So our formula for the weight is

$$\varphi_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{c}{\phi(e^{i\lambda})\phi(e^{-i\lambda})} e^{ij\lambda}\, d\lambda.$$

Of course we can write $\phi(B)|\zeta|^2 = (B - \zeta)(B - \overline{\zeta})$; here we let ζ denote the root in the upper half of the complex plane (so ζ is in the first quadrant and $\overline{\zeta}$ lies in the fourth quadrant). Since the integral is a rational function in $e^{i\lambda}$, we use the standard technique (Henrici, 1974, p. 249-250) of substituting $z = e^{i\lambda}$. Letting γ denote the unit circle in the complex plane, we have

$$\begin{aligned}\varphi_j &= \frac{1}{2\pi i} \oint_\gamma \frac{c|\zeta|^4}{(z-\zeta)(z-\overline{\zeta})(z^{-1}-\zeta)(z^{-1}-\overline{\zeta})} z^{j-1}\, dz \\ &= \frac{1}{2\pi i} \oint_\gamma \frac{c|\zeta|^2}{(z-\zeta)(z-\overline{\zeta})(z-\zeta^{-1})(z-\overline{\zeta}^{-1})} z^{j+1}\, dz\end{aligned}$$

by change of variable ($dz = i\, z\, d\lambda$) and simplification. Now since $j \geq 0$, the only poles of the integrand are at ζ, $\overline{\zeta}$, ζ^{-1}, and $\overline{\zeta}^{-1}$. The roots are distinct so long as $q > 0$ – see (13) – which is always the case; therefore the poles are all simple. By the Cauchy integral formula (Henrici, 1974, Theorem 4.7*b*), we can compute the integral by summing the residues of the integrand over poles within γ; since ζ, $\overline{\zeta}$ are outside the unit circle, this means the relevant poles are ζ^{-1} and $\overline{\zeta}^{-1}$. Thus

$$\begin{aligned}\varphi_j &= \frac{c|\zeta|^2 z^{j+1}}{(z-\zeta)(z-\overline{\zeta})(z-\overline{\zeta}^{-1})}\Big|_{z=\zeta^{-1}} + \frac{c|\zeta|^2 z^{j+1}}{(z-\zeta)(z-\overline{\zeta})(z-\zeta^{-1})}\Big|_{z=\overline{\zeta}^{-1}} \\ &= \frac{c|\zeta|^2 \zeta^{-(j+1)}}{(\zeta^{-1}-\zeta)(\zeta^{-1}-\overline{\zeta})(\zeta^{-1}-\overline{\zeta}^{-1})} + \frac{c|\zeta|^2 \overline{\zeta}^{-(j+1)}}{(\overline{\zeta}^{-1}-\zeta)(\overline{\zeta}^{-1}-\overline{\zeta})(\overline{\zeta}^{-1}-\zeta^{-1})} \\ &= c\,|\zeta|^2\, 2\, Re \left(\frac{\zeta^{-(j+1)}}{(\zeta^{-1}-\zeta)(\zeta^{-1}-\overline{\zeta})(\zeta^{-1}-\overline{\zeta}^{-1})} \right),\end{aligned}$$

where Re denotes the real part. Now using (14) and (15), we can determine the other quantities in the formula for φ_j. The real part in the last expression for φ_j can be rewritten as

$$\begin{aligned}&\frac{\zeta^{-(j+1)}}{(\zeta^{-1}-\zeta)(\zeta^{-1}-\overline{\zeta})(\zeta^{-1}-\overline{\zeta}^{-1})} \\ &= \frac{\zeta^{-(j+1)}(\overline{\zeta}^{-1}-\overline{\zeta})(\overline{\zeta}^{-1}-\zeta)(\overline{\zeta}^{-1}-\zeta^{-1})}{(|\zeta|^{-2} - 2\cos 2\theta + |\zeta|^2)(|\zeta|^{-2} - 2 + |\zeta|^2)(2|\zeta|^{-2} - 2|\zeta|^{-2}\cos 2\theta)} \\ &= |\zeta|^{4-j}(1-|\zeta|^{-2}) \frac{|\zeta|^2 i \sin\theta e^{-i(j+1)\theta} - i \sin\theta e^{-i(j-1)\theta}}{(1-2\cos(2\theta)|\zeta|^2 + |\zeta|^4)(1-2|\zeta|^2+|\zeta|^4)(1-\cos(2\theta))}.\end{aligned}$$

Substituting and simplifying, we have the final formula for φ_j:

$$\varphi_j = \frac{2\, c\, |\zeta|^{4-j} \sin\theta \left(|\zeta|^2 \sin(j+1)\theta - \sin(j-1)\theta \right)}{(1-2\cos(2\theta)|\zeta|^2 + |\zeta|^4)(|\zeta|^2 - 1)(1-\cos(2\theta))}, \tag{19}$$

which is expressed entirely in terms of c, q, $|\zeta|$, and θ. From (19) we can see the geometric rate of decay of the coefficients, as well as the side lobe behavior introduced by the sine functions. From (15) we see that large values of q drive θ to $\pi/2$ at rate $q^{1/2}$, which will make the $\sin(j+1)\theta$ and $\sin(j-1)\theta$ terms larger in absolute value; thus there will be more "wiggle" in the coefficient sequence for larger values of q. Conversely, small values of q will generate less wiggle, or the appearance of greater width in the coefficient sequence. This makes sense, since a low SNR indicates that more smoothing is necessary, which in turn implies a need to positively weight more of the nearby data. Section 4 below provides some illustrations of these observations.

4 Illustrations

In the original paper of Hodrick and Prescott (1997), the authors consider the HP filter with $q = 1/1600$. In this section we focus on this choice of the SNR, demonstrating the applications of Section 3. By (10)-(12), the AR polynomial is

$$\phi(B) = 1 - 1.777091B + 0.7994438B^2$$

with $c = 0.0004996524$. The magnitude of the roots is 1.118423, and the angle θ is ± 0.1116866 radians. The period associated with θ is $\pi/(6\theta)$ expressed in years; for $q = 1/1600$, this becomes

$$\pi/(6\theta) \doteq 4.688107,$$

a reasonable cycle period (see Harvey and Trimbur (2003) for a discussion). Of course the exact dynamics of the implied cycle model will depend upon $|\zeta|$ and θ, as well as $\varphi(B)$ and $\theta(B)$ in the original model for Y_t.

Next, we consider the coefficients given by (19), plotting for $j = 0, 1, \cdots, 100$. We compare these exact values to the coefficients found in Hodrick and Prescott (1997):

$$w_j = c^j \left[a_1 \cos(b|j|) + a_2 \sin(b|j|)\right] \tag{20}$$
$$a_1 = 0.056168, \quad a_2 = 0.055833, \quad b = 0.11168, \quad c = 0.8941$$

These correspond to $q = 1/1600$, and agree with φ_j up to three decimal places. The discrepancy may be due to numerical error, assuming that numerical methods were used to generate the w_j (this is not described in Hodrick and Prescott (1997), but they refer to Miller (1946) instead). In Figure 2 we plot the exact coefficients φ_j; this general shape is familiar to most readers. In Figure 3 we plot the difference $\varphi_j - w_j$ for $0 \leq j \leq 100$, which gives an idea of the discrepancy. The vertical scale is in millionths for easier readability. These discrepancies are so minute, that they would hardly affect filter output for any real data examples. Still, it is pleasing to have an exact formula for the weights in terms of q. The computations of this section were produced using a short R program, which is available from the author upon request (tucker.s.mcelroy@census.gov).

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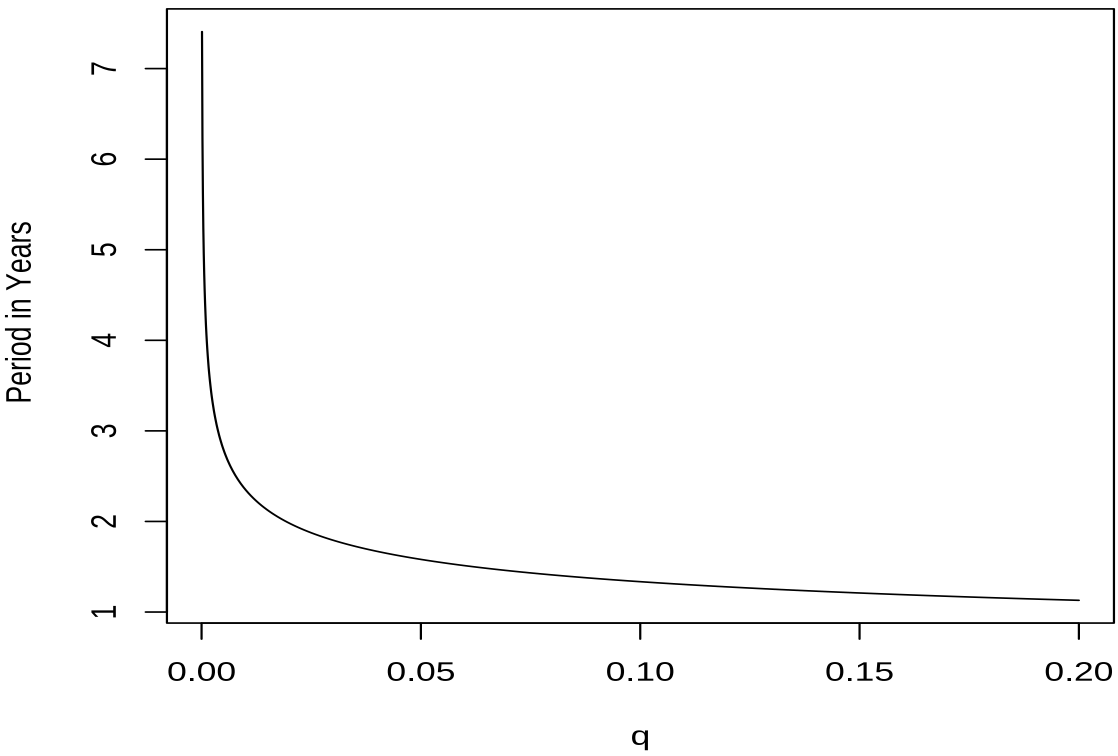

Figure 1: SNR q plotted against Cycle period (in years), for monthly data.

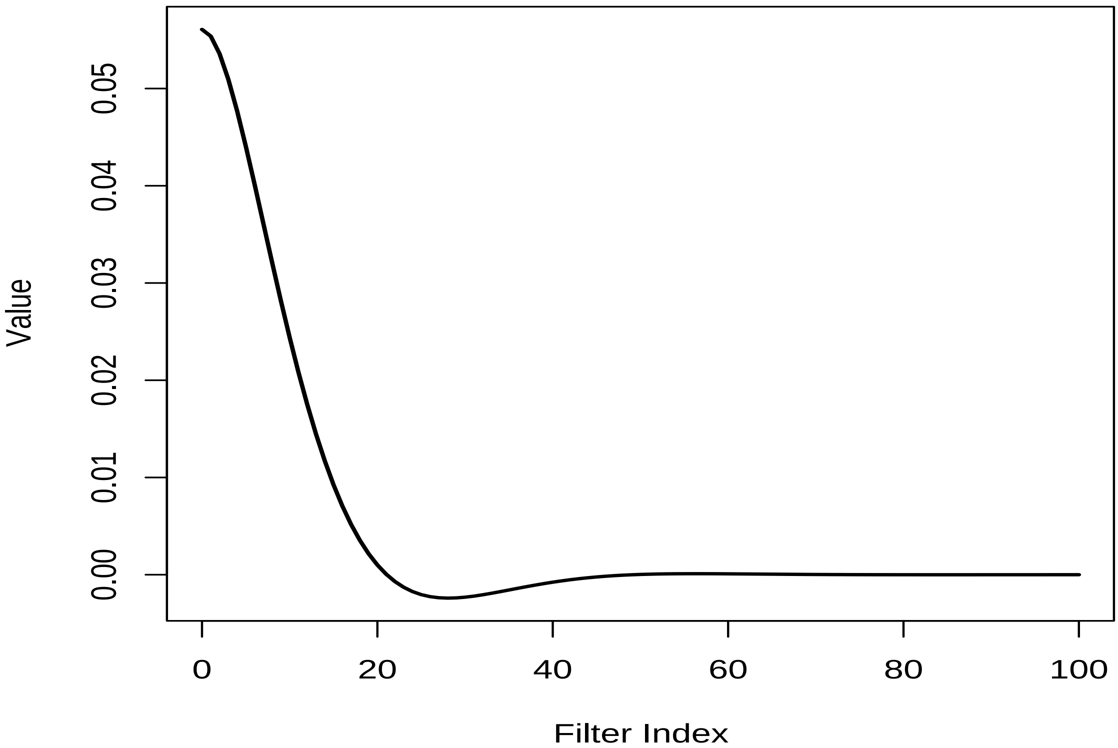

Figure 2: Exact weight coefficients for the HP filter.

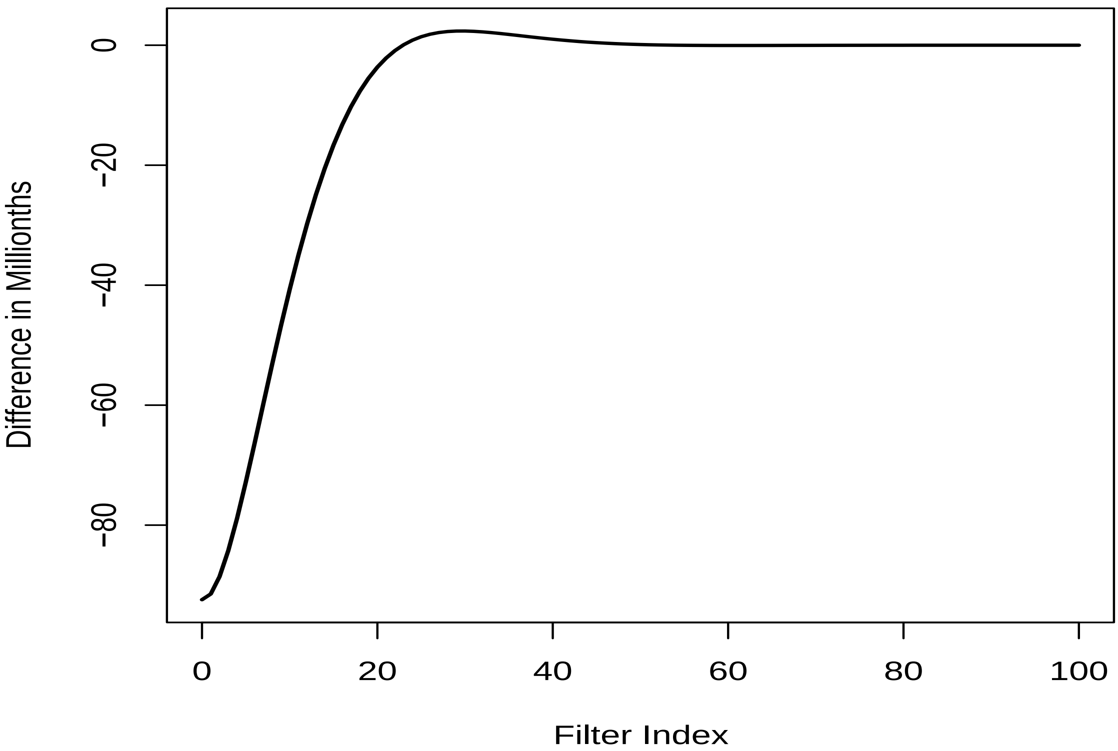

Figure 3: Difference of weights $\varphi_j - w_j$ for the HP filter. Vertical scale is in millionths.