Collateral Secured Loans in a Monetary Economy*

Leo Ferraris†
Universidad Carlos III de Madrid

Makoto Watanabe‡
Universidad Carlos III de Madrid

June 15, 2007

Abstract

This paper presents a microfounded model of money where durable assets serve as a guarantee to repay consumption loans. We establish steady state equilibria where money and bank credit coexist. In such an equilibrium, a larger investment in durable capital relaxes the borrowing constraint faced by consumers and thus provides a way to mitigate their costs of money holdings. We show that the occurrence of over-investment and the behavior of capital accumulation depend on the rate of inflation, relative risk aversion of agents and the marginal productivity of the capital goods.

Keywords: Collateral, Money, Search
JEL: E40

*We are grateful to Boragan Aruoba, Aleksander Berentsen, Ken Burdett, Gabriele Camera, Melvyn Coles, Miquel Faig, Nobuhiro Kiyotaki, Ricardo Lagos, Alberto Trejos, Neil Wallace, Christopher Waller, Randall Wright and participants at the Optimal Monetary Policy Conference in Ascona and SED 2007 annual meeting in Prague for useful comments and discussion throughout the course of this project. Financial support from the Spanish government in the form of research grant, SEJ 2006-11665-C02, and research fellowship, Juan de la Cierva, is gratefully acknowledged.

†Corresponding author. Department of Economics, Universidad Carlos III de Madrid, Calle Madrid 126, 28903 Getafe Madrid, SPAIN. Email: lferrari@eco.uc3m.es, Tel.: +34-91-624-9619, Fax: +34-91624-9329.

‡Department of Economics, Universidad Carlos III de Madrid, Calle Madrid 126, 28903 Getafe Madrid, SPAIN. Email: mwatanab@eco.uc3m.es, Tel.: +34-91624-9331, Fax: +34-91624-9329.

1 Introduction

Frictions are a necessary ingredient for money to emerge as a medium of exchange. Anonymity is essential. In a recent study, Berentsen, Camera and Waller (2006) establish a framework in which agents' anonymity is preserved on the goods market but not on the credit market, and bank credit plays a beneficial role. Anonymity may, however, impede the smooth working of credit systems, which are based on the exclusion of those who default from future access to loans. Lagos and Rocheteau (2006) study an economy where agents' anonymity is pervasive and trading arrangements based on capital goods, like cattle in primitive societies, can serve as a medium of exchange. In the present study, we take the view that markets are subject to frictions, and in particular, agents are anonymous in both good and credit deals. We present an alternative model in which anonymous agents can use capital goods, like real estate, as a guarantee of repayment of bank loans. We are motivated by the fact that still to the day, collateral secured loans account for a high percentage of all loans in industrialized countries.[1]

The environment we consider is a version of the divisible money framework developed in Lagos and Wright (2005).[2] In our economy a competitive banking system operates. Banks do not have monitoring or enforcement technologies. Agents can, however, obtain a bank loan by committing their capital asset as collateral and by signing a contract, which stipulates an amount of money, an interest payment, and the obligation to repay the loan. Banks have the ability to seize the committed capital if repayment does not happen. As the amount of borrowing is bounded by the level of capital they hold, agents are subject to a borrowing constraint of the form studied by Kiyotaki and Moore (1997) in a non-monetary economy.

Within this setup, accumulating capital over and above what it would be optimal from a purely productive point of view is a way for agents to relax their borrowing constraint. Two situations can arise as steady state monetary equilibria where money and bank credit coexist: one is where the borrowing constraint is not binding and capital is at the first best

[1]According to the last Federal Reserve Survey of Terms of Business Lending released on September 19th in 2006, the value of all commercial and industrial loans secured by collateral made by US banks accounted for 46.9 percent of the total value of loans in the US. Especially for commercial loans, the typical asset used as collateral is real estate. In 2004, 47.9 percent of the US households had home-secured debt, whereby their house was used as a guarantee of repayment (Survey of Consumer Finances, Board of Governors of the Federal Reserve System, 2004).

[2]The present model has features in common with Berentsen, Camera and Waller (2006), which introduces bank credit, and Aruoba and Wright (2003), which introduces capital accumulation in the Lagos and Wright model. We discuss these and other related works at the end of Section 3.

level, whereas the other is where the borrowing constraint is binding and capital is above the first best level. Given the demand for bank credit, the tightness of borrowing constraint is determined by the amount of capital to use for a given production, and the constraint is likely to be binding when capital is relatively scarce.

As the contract with a bank is written in nominal term, a financial obligation toward the bank is free from real costs of inflation, thereby the demand for bank credit increases in response to inflation. In turn, the binding borrowing constraint constitutes a channel through which monetary growth affects capital accumulation. We show that the effect of inflation on capital investment decision is determined by the marginal net benefit of making a loan relative to that of money holdings. For example, capital accumulation is decreasing in the rate of inflation if relative risk aversion of agents is lower than one. In this case, the demand for consumption is more than unit elastic in inflation, hence agents reduce their need for capital as collateral for loans. The same logic applies to the other cases where relative risk aversion is equal to (greater than) one and capital accumulation is constant (increasing) in inflation. At low rates of inflation, we also found a possibility that scarce capital implies credit rationing for borrowers and yields zero nominal interest rate. In such a case, the borrowing constraint is strictly tight and agents always overinvest in capital as inflation grows, irrelevant of the relative risk aversion parameter.

The rest of the paper is organized as follows. Section 2 presents the model, derives the equilibrium and contains a discussion of the related literature. Section 3 concludes. All omitted proofs are contained in the Appendix.

2 The Model

2.1 The Environment

The model is built on a competitive version of Lagos and Wright (2005), where agents take price as given in each market we describe below. Time is discrete and continues forever. There is a $[0, 1]$ continuum of infinitely-lived agents. Each period is divided into two sub-periods, called day and night. A perfectly competitive market opens in each sub-period. Economic activity differs between day and night. During the day, agents can trade a perishable consumption good and face randomness in their preferences and production possibilities. An agent is a *buyer* with probability σ in which case he wants to consume but cannot produce, whereas an agent is a *seller* with probability $1-\sigma$ in which case he is able to produce but does not wish to consume.[3] During the night, agents can trade a durable good that can be used for consumption or investment. In contrast to the first sub-period, there is no randomness in the second sub-period, and all agents can produce and consume simultaneously.

There is an intrinsically worthless good, which is perfectly divisible and storable, called fiat money. We assume that all goods trades are anonymous and so trading histories of agents are private knowledge. Combined with the presence of randomness described above, anonymity in goods trades motivates an essential role of money: sellers must receive money for immediate compensation of their products. The supply of fiat money is controlled by the government so that $M = \pi M_{-1}$, where M denotes the money stock at a given period, M_{-1} the previous period, and π the gross growth rate of the money supply which we assume to be constant. New money is injected, or withdrawn, at the start of each period by lump-sum transfers or taxes at a rate denoted by τ. Both buyers and sellers receive these transfers, which amounts to τM_{-1}, equally.

Consumption during the day yields utility $u(q_b)$ that, we assume, satisfies $u'(\cdot) > 0,\ u''(\cdot) < 0,\ u'(0) = \infty$, and $u'(\infty) = 0$, where q_b represents the amount of daytime-consumption. Production during the day requires utility cost $c(q_s) = q_s$, where q_s represents the amount of daytime-production. Agents obtain utility from x - i.e. consumption of durable goods- during the night, given by $U(x) = x$.[4] Simultaneously, agents can produce these goods using capital k. We assume that one's capital is not mobile so that it cannot be carried into the day market. Agents have an access to a production technology $f(k)$ that satisfies $f'(\cdot) > 0,\ f''(\cdot) < 0,\ f'(0) = \infty$, and $f'(\infty) = 0$. In what follows, we study situations where capital is sustainable over time. That is, we study the range of capital satisfying $0 \leq k \leq k'$ where k' defines the the maximal sustainable level of capital and is a solution to $f(k') = k'$. Capital depreciates at a rate $\delta \in (0,1)$. Agents discount future payoffs at a rate $\beta \in (0,1)$ across periods, but we assume for simplicity that there is no discounting between the two sub-periods.

[3] This formulation is adopted also in Berentsen, Camera and Waller (2006), Lagos and Rocheteau (2005), and Rocheteau and Wright (2005).

[4] As is common in the divisible money models, the linear (or quasi linear) preferences of nighttime consumption is a device to make the distribution of money holdings degenerate at the beginning of a period, which in turn makes the model highly tractable.

There exist private competitive banks, accepting deposits and issuing loans. Each period, before entering the day market, but after having discovered whether they are going to be buyers or sellers, agents can contact a bank, in order to deposit their money or obtain a loan, denoted by d, or l, respectively. Loans are repaid and deposits are withdrawn during the night of the same period. As in goods trades, agents are anonymous in financial transactions and their credit histories are private knowledge. Banks do not have technologies that allow them to punish borrowers by excluding them from future financial transactions in case of default. The debt repayment, however, can be enforced by using collateral. Should they find themselves in need of a loan for daytime consumption, individual agents can commit part or all of their physical capital as a guarantee of repayment. The bank has the right to seize the collateral when the loan is not paid back, and thus voluntary repayment can be ensured.[5] Assuming that output is not verifiable, we focus our attention on credit deals where only capital can serve as collateral. Assuming further that individual agents have inferior skills in verifying and seizing assets relative to banks and capital is not mobile from the location where it was produced implies that promises backed by collateralized capital cannot circulate as credit among individuals during the day.[6] Hence, fiat money is still used as a medium of exchange in goods trades.

2.2 The Social Optimum

We shall begin with the first best solution. The social planner treats agents symmetrically and maximizes average expected utility. The planner's problem describes:

$$J(k) = \max_{q_b, q_s, k_{+1}, x} \left[\sigma u(q_b) - (1-\sigma) q_s + x + \beta J_{+1}(k_{+1})\right]$$

$$\text{s.t.} \quad \sigma q_b = (1-\sigma)\, q_s, \tag{1}$$

$$x = F(k) - k_{+1}, \tag{2}$$

where $F(k) \equiv (1-\delta)k + f(k)$. Eq. (1) is the feasibility constraint for daytime consumption, and (2) is the feasibility constraint for nighttime consumption. At night, the amount of durable goods available for consumption, x, is provided by the total of the undepreciated and the newly

[5] See Kiyotaki and Moore (1997) for an extensive discussion on the use of capital as collateral.

[6] We discuss later the issue that bank notes or bilateral credit backed by physical capital could circulate among private agents, which our model does not address explicitly.

produced, $F(k)$, minus the amount carried into the next period, k_{+1}.

The optimal solution in steady state, denoted by $q_b^*,\ q_s^*,\ k^*,\ x^*$, satisfy the following first order conditions:

$$u'\left(q_b^*\right) = 1, \tag{3}$$

$$\beta F'(k^*) = 1, \tag{4}$$

and (1) and (2): $q_s^* = \frac{\sigma}{1-\sigma} q_b^*$; $x^* = F(k^*) - k^*$. At the optimum, the marginal utility of consumption is set equal to the marginal cost of production during the day, while the marginal utility of consuming one unit of durable goods (= 1) at a given night is set equal to the discounted value of the marginal returns, accruing at the following night, from accumulating an extra unit of capital ($= \beta F'(k^*)$).

2.3 Steady-state equilibrium

In what follows, we construct symmetric and steady-state equilibria with money and credit ($\phi, l, d > 0$) where all agents take identical strategies and all real variables are constant over time. Before proceeding, it is worth mentioning the characteristics of credit deals in such equilibria. First, given our environment, it is straightforward to show that buyers will not deposit their money because they will be able to use it for consumption, while sellers will not want to borrow money because they have no use of it at day. Hence, in the credit transactions with banks buyers are borrowers while sellers are depositors each day. Second, because their capital is the only asset that can be used as collateral, individual buyers face an upper bound on the amount of borrowing if they wish to consume beyond their budget. Formally, if a buyer holds capital k at a given period, then he can borrow l in total at that day, as long as it satisfies

$$\phi(1 + i)l \leq k, \tag{5}$$

where ϕ is the value of a unit of money and i the nominal interest rate (yet to be determined on the competitive credit market). The L.H.S. of the above inequality represents the real-valued repayment of his debt. The repayment happens before capital for the next period is selected, thus the above inequality holds at any given period (independently of stationarity). An important point here is that the property of equilibrium depends on whether (5) is binding or not. When the constraint is binding, one prefers to borrow up to the maximum and so his capital holdings play dual roles, that is, one is to determine the marginal productivity of nighttime production and the other is to determine the budget set of daytime consumption as a buyer. The latter role of capital holdings is absent when the constraint is not binding.

Night market We work backward and start with the night market. As already mentioned, during the night, agents consume, produce, and trade durable goods, and clear the credit balances they have had with banks during the day. The expected value of an agent entering the night market at a given period with holding m of money, l of loans, d of deposits and k of capital, denoted by $W(m, l, d, k)$, satisfies

$$\begin{aligned} W(m,l,d,k) &= \max{}_{x,m_{+1},k_{+1}} \ [x + \beta V(m_{+1}, k_{+1})] \\ \text{s.t.} \ \ x + k_{+1} + \phi m_{+1} + \phi(1+i)\,l &= F(k) + \phi m + \phi(1+i)\,d \end{aligned} \tag{6}$$

where $V(m_{+1}, k_{+1})$ denotes the expected value of operating in the next day market with holding m_{+1} money and k_{+1} capital. The nominal price in the night market is normalized by 1, and so ϕ represents the relative price of money. If the agent has been a buyer during the day, then $d = 0$ and he consumes x units and repay $(1+i)l$ units of money by producing and selling $F(k)$ units and using m units of money he initially holds. If the agent has been a seller during the day, then $l = 0$ and he consumes x units by producing and selling $F(k)$ units and using m units of money he initial holds and $(1+i)d$ units of monetary repayment of his deposit. Note that banks are competitive so the interest rate is the same across loans and deposits. After the night market is closed, the agent carries forward m_{+1} money and k_{+1} capital to the following period.

Solving (6) for x and substituting it into the value function, the first order conditions with respect to m_{+1} and k_{+1} are respectively derived as follows.

$$\beta V_m(m_{+1}, k_{+1}) = \phi, \tag{7}$$

$$\beta V_k(m_{+1}, k_{+1}) = 1, \tag{8}$$

where $V_i \equiv \frac{\partial V(m,k)}{\partial i}$ for $i = m, k$. It is clear from these expressions that the m_{+1}, k_{+1} are determined independently of both m, k, and hence all agents hold the same amount of money and capital at the beginning of any given day market.[7]

Finally, the envelope conditions are:

$$W_m = \phi; \tag{9}$$

$$W_l = -\phi\,(1+i)\,; \tag{10}$$

$$W_d = \phi\,(1+i)\,; \tag{11}$$

$$W_k = F'(k), \tag{12}$$

where $W_i \equiv \frac{\partial W(m,l,d,k)}{\partial i}$ for $i = m, l, d, k$.

Day market Agents during the day either consume and borrow as buyers or produce and deposit as sellers. All agents start any given period with the same amount of money and capital holdings. The expected value of an agent, $V(m,k)$, entering the day market with m money and k capital, satisfies:

$$\begin{aligned} V(m,k) &= \sigma \left\{ \begin{array}{c} \max_{\,q_b,l}\ \left[u(q_b) + W(m + \tau M_{-1} + l - pq_b, l, d, k)\right] \\ \text{s.t. } pq_b \leq m + \tau M_{-1} + l \\ \phi(1+i)l \leq k \end{array} \right\} \\ &\quad +(1-\sigma) \left\{ \begin{array}{c} \max_{\,q_s,d}\ \left[-q_s + W(m + \tau M_{-1} - d + pq_s, l, d, k)\right] \\ \text{s.t. } d \leq m + \tau M_{-1} \end{array} \right\} \end{aligned}$$

where p is the nominal price of daytime goods. If the agent happens to be a buyer with probability σ, then he spends pq_b money for his consumption, which is no greater than his initial money m plus the monetary transfer τM_{-1} and a loan l that he takes out from a bank. His loan l is subject to the credit constraint given the amount of capital k he has accumulated from the previous night. If the agent happens to be a seller with probability $1-\sigma$, then he produces q_s units and obtains pq_s money. At the same time, he deposits d money which is no greater than his initial money m plus the monetary transfer τM_{-1}. The agent then moves on to the night market with the remaining money which differs across these events.

[7] Note, however, that the nighttime consumption x (determined to satisfy (6)) differs across agents depending on the credit status and the initial money holding at the beginning of night. See the Appendix for details.

The first order conditions are:

$$
\begin{aligned}
pW_m + p\lambda &= u'\left(q_b\right), && (13)\\
pW_m &= 1, && (14)\\
W_l + W_m &= \gamma\phi(1+i) - \lambda, && (15)\\
W_d - W_m &= \rho, && (16)
\end{aligned}
$$

where $\lambda \geq 0$ is the multiplier of the buyer's budget constraint, $\gamma \geq 0$ the multiplier of the credit constraint, $\rho \geq 0$ the multiplier of the seller's deposit constraint.

It is worth mentioning some properties of the optimal choices in the day-market that are immediate from the above conditions. First, (9) and (14) imply:

$$\frac{1}{p} = \phi. \tag{17}$$

That is, the seller produces up to the point where the marginal costs of production per unit of money at day ($= 1/p$) and at night ($= \phi$) are equal.

Second, (9), (10), (13), (14), (15) yield

$$u'(q_b) = 1 + \frac{\lambda}{\phi} = (1+\gamma)(1+i). \tag{18}$$

The first equality implies that, given $\phi > 0$, the complementary slackness condition for the buyer's budget constraint requires

$$\left[u'\left(q_b\right) - 1\right]\left[m + \tau M_{-1} + l - pq_b\right] = 0. \tag{19}$$

Similarly, the second equality implies that the complementary slackness condition for the credit constraint requires

$$\left[u'(q_b) - (1+i)\right]\left[k - \phi(1+i)l\right] = 0. \tag{20}$$

Observe that for $\gamma = 0$, we have $u'(q_b) = 1 + i$ and $k \geq \phi(1+i)l$ in which case the buyer borrows up to the point where the marginal benefit of an extra unit of loan ($= u'(q_b)$) equals the marginal cost ($= 1 + i$). For $\gamma > 0$, we have $u'(q_b) > 1 + i$ and $k = \phi(1+i)l$ in which case the credit constraint is binding and the marginal benefit of a loan exceeds its marginal cost.

Third, (9), (10), (16) yield $\phi i = \rho$, hence the complementary slackness condition for the seller's deposit constraint requires, given $\phi > 0$, that

$$i\left[m + \tau M_{-1} - d\right] = 0. \tag{21}$$

For $\rho > 0$, we must have $d = m + \tau M_{-1}$ and $i > 0$ in which case the seller has a strict incentive to deposit his money. For $\rho = 0$, we must have $d \leq m + \tau M_{-1}$ and $i = 0$ in which case the seller is indifferent between depositing and holding his money with him. In what follows, we make a tie-breaking assumption that sellers deposit their money, if indifferent to doing so, hence $d = m + \tau M_{-1}$ holds for any $i \geq 0$.[8]

Euler equations We now derive the Euler equations. Using (7), (9), (11), (13), (16), (17) and the envelope condition, $V_m(m, k) = W_m + \sigma\lambda + (1 - \sigma)\rho$, with an updating, we obtain the Euler equation for money holdings:

$$\phi = \beta\phi_{+1}\left[\sigma u'(q_{b,+1}) + (1 - \sigma)(1 + i_{+1})\right]. \tag{22}$$

In the above equation, the marginal cost of obtaining an extra unit of money today $(= \phi)$ equals the discounted value of its expected marginal benefit obtained tomorrow. The marginal value of money is the marginal utility $(= u'(\cdot))$ when a buyer, or an interest payment of an extra unit of deposit $(= 1 + i)$ when a seller.

Similarly, using (8), (10), (12), (13), (15), (17) and the envelope condition, $V_k(m, k) = W_k + \sigma\gamma$, with an updating, we obtain the Euler equation for capital holdings:

$$1 = \beta\left[\sigma\left(\frac{u'(q_{b,+1})}{1 + i_{+1}} - 1\right) + F'(k_{+1})\right] \tag{23}$$

where the marginal cost of accumulating an extra unit of capital today $(= 1)$ equals the discounted value of its expected marginal benefit accruing tomorrow. The benefit of capital consists of two parts. On the one hand, the agent obtains the marginal returns $(= F'(\cdot))$ for the nighttime production. On the other hand, if the agent turns out to be a buyer, then he will be able to borrow an extra amount of funds equal to $\frac{1}{1+i_{+1}}$, since the value of a unit of capital will have to be enough to repay the gross interest payment of his loan. This will generate the benefit of an additional loan given by the marginal utility of daytime consumption $(= u'(\cdot))$ minus the repayment cost $(= 1 + i_{+1})$. Clearly, the higher the net benefit of making a loan as a buyer $(= u'(q_{b,+1})/(1 + i_{+1}) - 1)$, the larger amount of capital investment the agent makes.

It is important to observe from (23) that the benefit of capital holdings (to increase the daytime consumption) generates a possibility that the growth rate of the value of money,

[8] This tie breaking assumption does not affect the qualitative nature of our equilibrium.

ϕ_{+1}/ϕ, affects the capital investment decision by individuals. To see this point, consider a case in which $u'(q_b) = 1 + i$ holds and so the credit constraint (5) is slack. In this case, (23) reduces to $1 = \beta F'(k_{+1})$ where the amount of capital holdings by individuals is determined independently of the money growth rate. When $u'(q_b) > 1 + i$, however, the credit constraint is binding and the level of consumption and capital holdings are jointly determined by (22) and (23). Hence, the binding credit constraint in our model provides a channel through which monetary policy, determining ϕ_{+1}/ϕ, can affect the individual decisions on both consumption and capital investment.

Market-clearing conditions So far, we have described the decision problems of a given individual agent taking the market prices p, ϕ, i as given. Each of the prices are determined by the respective market-clearing condition. These are the last requirements for symmetric and steady-state equilibria in our model. For the daytime goods, since all buyers buy q_b units and all sellers sell q_s units at any given period, the day-market clearing condition is given by

$$\sigma q_b = (1 - \sigma)\, q_s. \tag{24}$$

For the nighttime goods, note that the level of nighttime consumption is bound to differ across agents depending on their daytime activity, while the level of capital holdings is not. Hence, the night-market clearing condition is given by

$$X = F(k) - k_{+1} \tag{25}$$

where X denotes the aggregate nighttime consumption which can be reduced to an expression with x satisfying (6) averaged over buyers with σ and sellers with $1 - \sigma$.

For the loans and deposits, all the credit deals are made through the competitive banks. Note that given the credit constraint, it is possible that the interest rate $i \geq 0$ does not adjust to yield the demand-supply balancing. Hence, if there is excess supply in the credit market when $i = 0$, we assume that the banks can hold voluntary reserves. This is consistent with our earlier assumption that sellers always deposit their money holdings when indifferent to doing so. Given this possibility, the credit-market clearing condition becomes

$$\sigma l = (1 - \sigma)(1 - \mu) d \tag{26}$$

where $\mu \in [0, 1]$ represents the rate of bank reserves.

Existence, uniqueness and characterization of steady-state equilibrium We now solve for equilibrium. We focus on steady-state equilibria where the aggregate real money supply, given by ϕM, is constant over time. So, we have $\frac{\phi_{+1}}{\phi} = \frac{1}{\pi}$. Further, since $M = (1+\tau)M_{-1} = \pi M_{-1}$, the value of money decreases at a rate equal to the gross rate at which the government injects money into the economy. Below, we consider policies where $\pi \geq \beta$, and when $\pi = \beta$ (which is the Friedman rule) we only consider the limiting equilibrium as $\pi \to \beta$.

Definition 1 *A symmetric steady-state monetary equilibrium with collateralized bank credit defines a set of prices, $p, \phi > 0$, $i, \geq 0$, and quantities, $q_b, q_s, x, d, l, k > 0$, $\mu \geq 0$ that satisfies the budget constraint (6), the first order conditions (and the Euler equations) (17), (22), (23), the complementary slackness conditions (19), (20), (21), and the market-clearing conditions (24), (25), (26), where identical agents take identical strategies and all real variables are constant over time.*

Any steady state equilibrium requires $x > 0$ for all agents, although we have not imposed it. In order to guarantee this, we assume

$$F(k^*) > q_b^* + \max\{k^*, (1-\sigma)\, q_b^*\},$$

where q_b^*, k^* are the first best level of consumption and capital satisfying (3) and (4). This inequality in turn requires an appropriate scaling of the production (or the utility) function.

To solve for equilibrium, the following lemma provides a useful guideline.

Lemma 1 *If an equilibrium given in Definition 1 exists for $\pi > \beta$, then when $i = 0$, we must have $u'(q_b) > 1$ and the binding credit-constraint, i.e., $\phi l = k$ when $i = 0$. Further, the buyer's budget constraint at day is binding for any $i \geq 0$.*

Proof. Observe first that (18) implies $u'(q_b) = 1 + \frac{\lambda}{\phi} = 1 + \gamma$ for $i = 0$. Suppose now that $\lambda = \gamma = 0$ when $i = 0$. Then, $u'(q_b) = 1$ by (18). However, this contradicts to the equilibrium we construct with $\pi > \beta$, because $u'(q_b) = 1$ and $i = 0$ imply (22) requires $\pi = \phi/\phi_+ = \beta$. Hence, for $\pi > \beta$, we must have $\lambda, \gamma > 0$ (and hence $u'(q_b) > 1$ and $\phi l = k$) when $i = 0$. Given this result, the second claim in the lemma is immediate by noting that the complementary slackness condition (20) requires $u'(q_b) \geq 1 + i$, thus $u'(q_b) > 1$ for any $i \geq 0$. This implies $\lambda > 0$ for any $i \geq 0$. ■

By (19) and (21), the binding budget constraints for buyers and sellers imply

$$pq_b = l + d. \tag{27}$$

Lemma 1 shows this equation holds for any $i \geq 0$. Further, when $i = 0$, buyers have a strict incentive to borrow and Lemma 1 shows the credit constraint is binding for any $\pi > \beta$. As already mentioned, there is a possibility of an excess supply in credit market in this case. Applying the binding credit constraint with $i = 0$ (i.e. $\phi l = k$) and (27) to (26) yields

$$\mu = 1 - \frac{\sigma k}{(1-\sigma)(q_b - k)} \geq 0$$

which implies that $\mu > 0$ when $i = 0$ and $k < (1-\sigma)q_b$. That is, if capital is scarce, the total amount of loans buyers make is strictly below the market clearing level at $i = 0$. The resulting idle deposits are held by banks as voluntary reserves and so $\mu > 0$. Note, however, that when $i > 0$ the banks' holding of a positive fraction of deposits (i.e., the banks' not lending out all deposits) cannot be part of an equilibrium given the competitive nature of the banking system. Therefore, irrelevant of whether the credit constraint is binding or not, we must have $\mu = 0$ when $i > 0$.

In sum, there are three possible cases for equilibrium: [1] an equilibrium without binding credit-constraint and with $i > 0$; [2] an equilibrium with binding credit-constraint and $i > 0$; [3] an equilibrium with binding credit-constraint and $i = 0$. In the last two cases monetary policies can impact on capital accumulations while in the first case they cannot. In the following propositions, we show that either type of equilibrium can emerge, depending on the coefficient of risk aversion, denoted by $\alpha = \alpha(q_b) \equiv -\frac{u''(q_b)}{u'(q_b)} q_b$, and on the first best level of capital $k^* = F'^{-1}(1/\beta)$ relative to σ and $q_b^* = u'^{-1}(1)$.

Proposition 1 *Suppose $\alpha < 1$. (A) If $k^* \geq (1-\sigma)\, q_b^*$, there is a unique equilibrium with unconstrained credit for any $\pi \in (\beta, \infty)$. (B) If $k^* < (1-\sigma)\, q_b^*$, there exist two critical levels of inflation rate, denoted by $\underline{\pi}$ and $\hat{\pi}$, such that a unique equilibrium exists: (i) with constrained credit and $i = 0$ for $\pi \in (\beta, \underline{\pi})$; (ii) with constrained credit and $i > 0$ for $\pi \in [\underline{\pi}, \hat{\pi}]$; (iii) with unconstrained credit for $\pi \in (\hat{\pi}, \infty)$ given $\lim_{q_b \to 0} u'(q_b) q_b < k^*/(1-\sigma)$.*

Proposition 2 *Suppose $\alpha = 1$. (A) If $k^* > (1-\sigma)\, q_b^*$, there exists a unique equilibrium with unconstrained credit for any $\pi \in (\beta, \infty)$. If $k^* = (1-\sigma)\, q_b^*$, a unique equilibrium exists with constrained credit and $i > 0$ for any $\pi \in (\beta, \infty)$. (B) If $k^* < (1-\sigma)\, q_b^*$, there exists a critical rate, denoted by $\underline{\pi}'$, such that a unique equilibrium exists with constrained credit and: (i) $i = 0$ for $\pi \in (\beta, \underline{\pi}')$; (ii) $i > 0$ for $\pi \in [\underline{\pi}', \infty)$.*

Proposition 3 *Suppose* $\alpha > 1$. *(A) If* $k^* > (1-\sigma)\, q_b^*$*, there exist two critical rates, denoted by* $\hat{\pi}'$ *and* $\tilde{\pi}'$*, such that a unique equilibrium exists: (i) with unconstrained credit for* $\pi \in (\beta, \hat{\pi}')$*; (ii) with constrained credit and* $i > 0$ *for* $\pi \in [\hat{\pi}', \tilde{\pi}']$*, given* $\lim_{q_b \to 0} u'(q_b) q_b > k^*/(1-\sigma)$*. (B) If* $k^* = (1-\sigma) q_b^*$*, a unique equilibrium exists with constrained credit and* $i > 0$ *for* $\pi \in (\beta, \widetilde{\pi}'']$*. If* $k^* < (1-\sigma)\, q_b^*$*, there exist two critical values, denoted by* $\underline{\pi}''$ *and* $\tilde{\pi}''$*, such that a unique equilibrium exists with constrained credit and: (i)* $i = 0$ *for* $\pi \in (\beta, \underline{\pi}'']$*; (ii)* $i > 0$ *for* $\pi \in (\underline{\pi}'', \tilde{\pi}'']$*. In each case, for* $\pi > \tilde{\pi}'$*,* $\pi > \widetilde{\pi}''$ *or* $\pi > \tilde{\pi}''$*, an equilibrium may not exist.*

Figure 1-6 provide a graphical representation of the equilibria established in Proposition 1-3. The following proposition summarizes the corresponding behavior of capital accumulation.

Proposition 4 *1. Suppose* $\alpha < 1$. *(A) If* $k^* \geq (1-\sigma)\, q_b^*$*, the level of capital* k *is constant at the first best* k^* *for all* $\pi \in (\beta, \infty)$*. (B) If* $k^* < (1-\sigma)\, q_b^*$*,* k *is increasing in* $\pi \in (\beta, \underline{\pi})$*, decreasing in* $\pi \in [\underline{\pi}, \hat{\pi})$ *and constant in* $\pi \in [\hat{\pi}, \infty)$*.*

2. Suppose $\alpha = 1$. *(A) If* $k^* \geq (1-\sigma)\, q_b^*$*,* k *is constant at* k^* *for all* $\pi \in (\beta, \infty)$*. (B) If* $k^* < (1-\sigma)\, q_b^*$*,* k *is increasing in* $\pi \in (\beta, \underline{\pi}')$ *and constant in* $\pi \in [\underline{\pi}', \infty)$*.*

3. Suppose $\alpha > 1$. *(A) If* $k^* > (1-\sigma)\, q_b^*$*,* k *is constant in* $\pi \in (\beta, \hat{\pi}')$ *and increasing in* $\pi \in [\hat{\pi}', \tilde{\pi}')$*. (B) If* $k^* \leq (1-\sigma)\, q_b^*$*,* k *is increasing in* $\pi \in (\beta, \widetilde{\pi}'')$ *or in* $\pi \in (\beta, \tilde{\pi}'')$*.*

Figure 1 depicts a case in which inflation does not affect capital accumulation for all $\pi > \beta$. In the other cases depicted in Figure 2-6, accumulation of capital is affected by inflation and thus by monetary growth for some range of the inflation rates. Essentially, accumulating capital over and above its first best level is a way for agents to relax their borrowing constraint, when the first best level itself is not abundant enough to perform fully both its productive and its collateral role. In turn, when capital is above the first best, monetary growth can affect its accumulation.

There are two margins that determine the type of equilibria and the effect of inflation. In the first margin, production technologies determine the level of capital that can be used for collateral, and the tightness of borrowing constraint given the level of consumption. When the marginal product of capital $F'(\cdot)$ is high, capital itself is at a low level. Hence, it is more likely that buyers face the binding credit constraint when k^* $(= F'^{-1}(1/\beta))$ is low (as shown in Figure 2, 3, 6) than when k^* is high (as shown in Figure 1, 4, 5). In all the former cases, observe that an equilibrium arises at low inflation rates with $i = 0$, because at low rates of inflation the real value of money balances and consumption are at high levels, which makes the borrowing constraint strictly tight when capital is scarce.

In the second margin, the benefit of an extra loan net of repayment costs, given by $u'(q_b)/(1 + i) - 1$, determines the buyers' demands of collateral loans, and the tightness of borrowing constraint given the level of capital. To see this, consider first the equilibrium with $i = 0$. Within this region, the repayment rate of a loan is fixed at $1 + i = 1$ while the cost of holding money increases in response to inflation. This implies that the net benefit of an extra unit of loan is given by $u'(q_b) - 1$ and increases in response to inflation, thereby when $i = 0$ agents accumulate more capital and obtain a larger fraction of loans as the rate of inflation increases.[9] As the total amount of loans gets sufficiently large due to the abundance of capital, the other type of equilibrium occurs at some inflation rates where there are no idle deposits held by banks and the credit market clears at $i > 0$.

When there is no excess supply in credit market, the interest rate adjusts to balance demand and supply. In this situation, inflation leads to an increase in $i > 0$ which in turn raises buyers' repayment cost of making a loan. Hence, an occurrence of the binding credit constraint and behavior of capital reflect the behavior of marginal costs and benefits of bank loans, relative to that of money holdings. The elasticity of day-time demand to inflation captures this margin, when $i > 0$, and so we build our intuition using the following property.

Corollary 1 *When $i > 0$, the elasticity of daytime consumption with respect to the rate of inflation, denoted by $\alpha_\pi \equiv -\frac{dq_b/d\pi}{q_b/\pi}$, satisfies: $\alpha_\pi \gtreqless 1$ if and only if $\alpha \lesseqgtr 1$.*

Given $i > 0$, Corollary 1 establishes a one-to-one relationship between the response of day-time demand to inflation and the relative risk aversion of agents, α. When agents are risk averse, i.e. when $\alpha < 1$, the demand for day-time consumption is more than unit elastic in π. In this case, capital accumulation cannot increase in inflation since agents reduce their demand for the day-time good more than one to one with inflation and thus do not need to accumulate extra capital for collateral. If capital is abundant, the credit constraint is never

[9]The ratio of bank loans to money holdings of buyers, given by $\psi \equiv l/(m + \tau M_{-1}) = (1 - \sigma)(1 - \mu)/\sigma$, increases in π when $\mu > 0$. When the credit market clears (i.e., when $\mu = 0$), $\psi = (1 - \sigma)/\sigma$ for all π.

binding for all π (Figure 1). If capital is relatively scarce, an equilibrium is with the binding credit constraint and $i > 0$ for moderate inflation rates, where the net benefit of collateral loan and capital accumulation decreases in inflation (Figure 2). As inflation grows further, capital reverts to its first best level and the credit constraint becomes not binding. When $\alpha = 1$, demand is unit elastic in inflation and agents reduce consumption one to one with inflation. In this case, if capital is abundant, agents are credit unconstrained for all π (Figure 1), and if capital is scarce, moderate and high inflation rates yield an equilibrium with just binding credit constraint and $i > 0$, where inflation does not alter the tightness of credit constraint and so capital remains constant at a high level (Figure 3). When $\alpha > 1$, demand is less than unit elastic in inflation. In this case, capital accumulation cannot decrease in inflation since agents reduce their demand for the day-time good less than one to one with inflation and thus need to accumulate extra capital for collateral to relax a tighter credit constraint. If capital is abundant, an equilibrium is with binding credit constraint and $i > 0$ at moderate inflation rates where capital increases in response to inflation. Eventually the economy reaches a point where such a high level of capital cannot be sustained and the credit equilibrium disappears (Figure 4). If capital is scarce agents start accumulating extra capital even at relatively low rates of inflation (Figure 5, 6).

2.4 Discussion

The main assumptions of the model are limited enforcement of contracts, anonymity and impossibility to monitor agents and the fact that capital is observable and verifiable. In our framework, they imply that agents can always refuse to work, agents can walk away with whatever output they produced, but their asset can be seized by the bank. Our assumptions about enforcement differ from those in Berentsen, Camera and Waller (2006), who introduce lending and borrowing - but not capital- in a Lagos and Wright (2005) framework. They assume that the bank can either enforce contracts perfectly or it can exclude agents from borrowing and lending for ever should they default once. In our paper enforcement is more limited than in their world. One of the crucial assumptions of the model is that capital cannot be moved. This assumption is adopted in Aruoba and Wright (2003), but not in the related paper by Aruoba, Waller and Wright (2006). These papers are concerned with the neoclassical dicothomy between nominal and real variables and specifically between money growth and capital accumulation. The former induces such a dicothomy assuming that capital cannot be moved and used in the day-time market, while the latter breaks the dicothomy assuming that capital has a cost saving role during day-time production. Our paper has both dicothomous and non-dicothomous regions, while maintaining throughout that capital cannot be moved from the night-time market place. This assumption, while capturing a realistic feature of many assets- especially real estate-, is necessary to exclude the possibility for capital to compete and possibly replace money as a means of exchange. Lagos and Rocheteau (2006) consider a world where capital competes with money as a means of exchange. A further assumption we make is that banks have superior skills in verifying and seizing assets, relative to dispersed individuals- who, we assume, don't have such skills-. This assumption, while realistic and common in the banking literature - see e.g. Diamond and Rajan (2001)- deserves some extra care in our framework. We saw that capital cannot be used as a medium of exchange, since it cannot be moved. However it could be used to guarantee bilateral promises agents may issue when meeting each other during the day, thus making money and bank loans redundant. If agents cannot seize assets, this never happens. It is interesting, though, to know what would it happen if agents themselves could seize assets. Here is an argument - along the lines of Lagos and Rocheteau (2006)- which answers such a question. When the First Best level of capital is enough to conduct trade on the day-time market so as to produce and consume the efficient quantity, then money is not useful and can be beneficially replaced by bilateral promises backed by capital. If the First Best level of capital is not enough, though, agents will over-accumulate it in order to use it as collateral for bilateral loans. In such a situation, trading with money and bank loans can reduce the over-accumulation of capital. This is true, obviously, for the region of the parameters space where over-accumulation doesn't happen. But even when over-accumulation does happen in equilibrium, capital is over-accumulated less in a monetary economy than in a non monetary one, since money allows agents to economize on it. Thus, if capital is relatively scarce, there exist a monetary equilibrium with bank loans which dominates the non-monetary equilibrium with bilateral collateralized loans. Such a conclusion also holds in case banks themselves start issuing private notes backed by collateral.

3 Conclusion

We considered an economy with lending and capital accumulation where capital can serve as collateral for consumption loans. We found scenarios where inflation affects capital accumulation and agents accumulate capital over and above its First Best level.

We believe the current model would be a fruitful framework to address, in future research, the question of the role played by the price of the durable good in explaining the persistence and amplification of monetary shocks.

References

[1] Aruoba, S. Boragan and Randall Wright (2003), Search, Money and Capital: a Neoclassical Dichotomy, *Journal of Money, Credit and Banking,* 35(6), pp.1086-1105.

[2] Aruoba, S. Boragan, Christopher Waller and Randall Wright (2006), Money and Capital, mimeo.

[3] Berentsen, Aleksander, Gabriele Camera and Christopher Waller (2006), Money, Credit and Banking, *Journal of Economic Theory,* forthcoming.

[4] Douglas, W. Diamond, and Raghuram G. Rajan (2001), Liquidity Risk, Liquidity Creation and Financial Fragility: A Theory of Banking, *Journal of Political Economy,* 109, pp.287-327.

[5] Kiyotaki, Nobuhiro and John Moore (1997), Credit Cycles, *Journal of Political Economy,* 105, pp.211-248.

[6] Lagos, Ricardo and Guillaume Rocheteau (2006), Money and Capital as Competing Media of Exchange, *Journal of Economic Theory,* forthcoming.

[7] ——————————— (2005), Inflation, Output, and Welfare, *International Economic Review,* 46, pp.495-522.

[8] Lagos, Ricardo and Randall Wright (2005), A Unified Framework for Monetary Theory and Policy Analysis, *Journal of Political Economy,* 113, pp.463-484.

[9] Rocheteau, Guillaume, and Randall Wright (2005), Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium, *Econometrica,* 73, pp.175-202.

4 Appendix

4.1 Proof of Proposition 1, 2, 3

In the main text, we have shown that (6), (17), (19), (20), (21), (22), (23), (24), (25), (26) are the equilibrium requirements in our economy. All that remains here is to find a solution $q_b,\ k,\ q_s,\ x,\ d,\ ,l,\ p,\ \phi > 0,\ i,\ \mu \geq 0$ to these equations. The equilibrium system can be reduced to the following equations that determine q_b, k, i, μ:

$$\frac{\pi}{\beta} = \sigma u'(q_b) + (1-\sigma)(1+i); \tag{28}$$

$$\frac{1}{\beta} = \sigma\left(\frac{u'(q_b)}{1+i} - 1\right) + F'(k); \tag{29}$$

$$\left[u'(q_b) - (1+i)\right]\left[k - \frac{(1-\sigma)(1-\mu)}{\sigma + (1-\sigma)(1-\mu)}(1+i)\,q_b\right] = 0; \tag{30}$$

$$\mu = \begin{cases} 1 - \frac{\sigma k}{(1-\sigma)(q_b - k)} & \text{iff } i = 0 \text{ and } k < (1-\sigma)q_b \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

To derive these equations, we use (17), (19), (20), (21), (22), (23), (26). In what follows, we first show the existence and uniqueness of $q_b, k > 0,\ i, \mu \geq 0$ to (28)-(31). There are six cases, depending on the coefficient of relative risk aversion, denoted by $\alpha \equiv -u''(q_b)q_b/u'(q_b)$, and on the efficient level of capital $k^* = F'^{-1}(1/\beta)$ relative to σ and $q_b^* = u'^{-1}(1)$. We examine each case in separation below. Given this solution, the equilibrium solution of other variables $q_s,\ x,\ d,\ ,l,\ p,\ \phi > 0$ is then identified by using (6), (24), (25), (26). This solution satisfies (6), (17), (19), (20), (21), (22), (23), (24), (25), (26) and so describes equilibrium.

<u>**Case 1-A:** $\alpha < 1$ **and** $k^* \geq (1-\sigma)q_b^*$**.**</u> For any $\pi \in (\beta, \infty)$, an equilibrium is without binding credit-constraint, exists, is unique and satisfies: $q_b \in (0, q_b^*)$, $k = k^*$, $i \in (0, \infty)$, $\mu = 0$, $x \in (0, \infty)$, $d \in (0, \infty)$, $l \in (0, \infty)$, $p \in (0, \infty)$, $\phi \in (0, \infty)$, $q_s \in (0, q_s^*)$.

Proof of Case 1-A. First of all, note that because equilibrium requires $u'(q) \geq 1+i$, (28) and (29) imply that: $q_b \to q_b^*$, $i \to 0$, $k = k^*$ as $\pi \to \beta$. If $k^* > (1-\sigma)q_b^*$, this further implies that, an equilibrium, if it exists for π close to β, must be without binding credit-constraint, and $\mu = 0$ and $q_b, i, k > 0$ satisfy:

$$u'(q_b) = \tfrac{\pi}{\beta}; \tag{32}$$

$$1 + i = \tfrac{\pi}{\beta}; \tag{33}$$

$$F'(k) = \tfrac{1}{\beta}. \tag{34}$$

Second, observe that: (i) $u'(q_b)\,q_b$ is strictly increasing in q_b when $\alpha \equiv -u''(\cdot)q_b/u'(\cdot) < 1$; (ii) q_b is strictly decreasing in π when (32) holds. Hence, when the credit constraint is not binding and $\alpha < 1$, the total amount of debt payment, given by

$$\phi(1+i)l = (1-\sigma)(1+i)q_b = (1-\sigma)u'(q_b)q_b,$$

is strictly decreasing in $\pi > \beta$. Because k is independent of π (in (34)), this implies that $k = k^* > (1-\sigma)q_b^* > (1-\sigma)u'(q_b)q_b$ hold and the credit constraint is not binding for all $\pi > \beta$, where equilibrium must satisfy (32)-(34). A solution to these equations exists and is unique, given our assumptions on $u(\cdot)$ and $f(\cdot)$. A similar procedure applies to obtain the solution for $k^* = (1-\sigma)q_b^*$, because the credit constraint must not be binding for all $\pi > \beta$, when $\alpha < 1$ and $k^* = (1-\sigma)q_b^*$. Given this solution, noting equilibrium satisfies $d = m + \tau M_{-1} = M$,

$\phi = 1/p$, $l = (1-\sigma)\, pq_b$, $(1-\sigma)d = \sigma l$ and $(1-\sigma)q_s = \sigma q_b$, implies that a solution for $\phi,\ p,\ d,\ l,\ q_s > 0$ exists and is unique.

Finally, given these equilibrium values, we identify the equilibrium value of nighttime consumption x. Eq. (6) implies for an agent who has been a buyer during the day, it holds that:

$$x = F(k) - k - \phi m_{+1} - \phi(1+i)l = F(k) - k - \sigma q_b - (1-\sigma)u'(q_b)q_b > F(k^*) - k^* - q_b^* > 0.$$

where the first equalities follow from the fact that the buyer does not carry money when entering the night market and $\phi m_+ = \phi d = \sigma q_b$, and the last two inequalities follow from $k = k^*$, $q_b^* > u'(q_b)q_b$, and our assumption that $F(k^*) > k^* + q_b^*$. Similarly, an agent who has been a seller has q_s money at the start of the night market and so

$$x = F(k) - k - \phi m_{+1} + q_s + \phi(1+i)d = F(k) - k - \sigma q_b + q_s + \sigma u'(q_b)q_b > F(k^*) - k^* - q_b^* > 0.$$

Therefore, the equilibrium exists and is unique.[10] ■

Case 1-B: $\alpha < 1$ **and** $k^* < (1-\sigma)q_b^*$**.** An equilibrium exists, is unique and implies that: the credit constraint is binding with $i = 0$ for $\pi \in (\beta, \underline{\pi})$; the credit constraint is binding with $i > 0$ for $\pi \in [\underline{\pi}, \hat{\pi}]$; the credit constraint is not binding for $\pi \in (\hat{\pi}, \infty)$, given $\lim_{q_b \to 0} u'(q_b)q_b < k^*/(1-\sigma)$.

Proof of Case 1-B. Observe first from (28) and (29) that $q_b \to 0$, $i \to \infty$ as $\pi \to \infty$, and hence from (29) and (30) that $u'(q_b)/(1+i) \to 1$ as $\pi \to \infty$ given $\lim_{q_b \to 0} u'(q_b)q_b < k^*/(1-\sigma)$. This implies that if an equilibrium exists for a sufficiently large π, then it must be without binding credit-constraint. Hence, given $k^* = k < (1-\sigma)q_b^* = (1-\sigma)u'(q_b)q_b$ around π close to β and $u'(q_b)q_b$ is strictly decreasing in π (when the constraint is not binding and $\alpha < 1$), there exists a unique cutoff value, denoted by $\hat{\pi} \in (\beta, \infty)$, that solves

$$k^* = k = (1-\sigma)u'(q_b)q_b \tag{35}$$

such that an equilibrium is without binding credit-constraint for $\pi > \hat{\pi}$ and is with binding credit-constraint for $\pi \leq \hat{\pi}$. As shown in the proof of Case 1-A, $\mu = 0$ and $q_b,\ k,\ i > 0$ are a unique solution to (32)-(34) when the constraint is not binding, and so for $\pi \in (\hat{\pi}, \infty)$ an equilibrium exists and is unique given $F(k^*) > k^* + q_b^*$.

For $\pi \leq \widehat{\pi}$, given $k^* < (1-\sigma)q^*$ it is possible that $i = 0$ and $\mu > 0$. Indeed, when $\pi \to \beta$, (28)-(31) imply $q \to q^*$, $k \to k^*$, $i \to 0$, $\mu \to \mu^* \equiv 1 - \sigma k^*/(1-\sigma)(q_b^* - k^*) > 0$. Hence, for π close to β, if an equilibrium exists then it must satisfy $i = 0$ and $q_b,\ k,\ \mu > 0$ that are given by

$$u'(q_b) = \tfrac{\pi - \beta(1-\sigma)}{\beta\sigma}, \tag{36}$$

$$F'(k) = \tfrac{1-[\pi-\beta]}{\beta}, \tag{37}$$

$$\mu = 1 - \tfrac{\sigma k}{(1-\sigma)(q_b - k)}. \tag{38}$$

Denoting by $\underline{\pi} \in (\beta, \hat{\pi})$ a unique solution to $\underline{k} = (1-\sigma)\underline{q}_b$ (which leads to $\mu = 0$), for $\pi \in (\beta, \underline{\pi}]$ a solution $q_b \in [\underline{q}_b, q_b^*)$, $k \in (k^*, \underline{k}]$, $\mu \in [0, \mu^*)$ to (36)-(38) exists and is unique. Given this solution, the other equilibrium values are uniquely identified by $d = M > 0$,

[10] Note that nighttime consumption is always larger for an agent who has been a seller than a buyer, and so $x > 0$ for a buyer also implies a non-negative nighttime consumption for a seller. Further, once we pin down $x > 0$ for a buyer, the corresponding value for a seller can also be identified by the night-market clearing condition (25). For this reason, in what follows, we only present the proof of $x > 0$ for a buyer.

$\sigma l = (1-\mu)(1-\sigma)d > 0$, $(1-\sigma)(1-\mu)pq_b = l\{\sigma + (1-\sigma)(1-\mu)\} > 0$, $\phi = 1/p > 0$, $(1-\sigma)q_s = \sigma q_b > 0$ and

$$x = F(k) - k - \phi m_+ - \phi l = F(k) - k - q_b > 0,$$

which follows from $k^* < k < (1-\sigma)q_b^*$ and our assumption that $F(k^*) > q_b^* + (1-\sigma)q_b^*$.

Observe above that μ is strictly decreasing in π and takes the minimum $\mu = 0$ at $\pi = \underline{\pi}$. This means, if an equilibrium exists for $\pi > \underline{\pi}$, then it must satisfy $\mu = 0$ and thereby $i > 0$ (whenever $k = (1-\sigma)(1+i)q_b$). Hence, for $\pi \in (\underline{\pi}, \hat{\pi})$ define:

$$\Phi(q_b, \pi) \equiv \left(\frac{\pi}{\beta} - \sigma u'(q_b)\right)\left(1 + \beta - \beta F'\left[\left(\frac{\pi}{\beta} - \sigma u'(q_b)\right) q_b\right]\right) - (1-\sigma)\pi = 0 \tag{39}$$

using (28) and (29). Observe that for $\pi \in (\underline{\pi}, \hat{\pi})$, $\Phi(\cdot)$ satisfies: $\Phi(\hat{q}_b;\ \pi) > 0$ where $\hat{q}_b \in (0, q_b^*)$ is given by $u'(\hat{q}_b) = \pi/\beta$; $\Phi(\underline{q}_b,\ \pi) < 0$ where $\underline{q}_b$ is given by $u'(\underline{q}_b) = (\pi - \beta(1-\sigma))/\beta\sigma$, and $\underline{\pi} \in (\beta, \hat{\pi})$ satisfies $1+\beta-\beta F'(\underline{k}) = \underline{\pi}$ and $\underline{k} = (1-\sigma)\underline{q}_b$. Therefore, because $\Phi(\cdot)$ is continuous in q_b and $\partial\Phi(\cdot)/\partial q_b > 0$, there exists a unique solution $q_b \in (\hat{q}_b, \underline{q}_b)$ that satisfies $\Phi(\cdot) = 0$ for $\pi \in (\underline{\pi}, \hat{\pi})$. Given $q_b > 0$ determined above, $k, i > 0$ solve for

$$k = \left(\tfrac{\pi}{\beta} - \sigma u'(q_b)\right) q_b, \tag{40}$$

$$1 + i = \tfrac{k}{(1-\sigma)q_b}, \tag{41}$$

which are obtained by applying $\mu = 0$ to (28) and (30). Note $k \in (k^*, \underline{k})$ and $i > 0$ satisfying (40) and (41) are both strictly increasing in $q_b \in (\hat{q}_b, \underline{q}_b)$ (given π), hence the solution exists and is unique. The other equilibrium values are uniquely identified by the same procedure as before, except that

$$x = F(k) - k - \sigma q_b - k > 0,$$

follows from $k^* < k < (1-\sigma)q_b^*$ and our assumption that $F(k^*) > q_b^* + (1-\sigma)q_b^*$. ■

Case 2-A: $\alpha = 1$ and $k^* \geq (1-\sigma)q_b^*$. For any $\pi \in (\beta, \infty)$, an equilibrium exists and is unique, without binding credit constraint if $k^* > (1-\sigma)q_b^*$, and with just binding credit constraint and $i > 0$ if $k^* = (1-\sigma)q_b^*$.

Proof of Case 2-A. The claim can be shown as in the proof of Case 1-A. That is, noting that (28) and (29) $q_b \to q_b^*$, $i \to 0$, $k \to k^*$ as $\pi \to \beta$, implies for $k^* > (1-\sigma)\,q_b^*$, an equilibrium, if it exists for π close to β, must be without binding credit-constraint (i.e., $k > (1-\sigma)(1+i)q_b$ holds) and $\mu = 0$ where q_b, i, k are determined by (32)-(34). Further, $\alpha = 1$ implies $u'(q_b)q_b$ is constant with respect to q_b, and so

$$k^* = k > (1-\sigma)q_b^* = (1-\sigma)u'(q_b)q_b,$$

for any $\pi \in (\beta, \infty)$. Hence, for any $\pi \in (\beta, \infty)$ the credit constraint is not binding and $\mu = 0$ and $q_b, i, k > 0$ are given uniquely by (32)-(34). Identifying the other equilibrium variables, which exist and are unique, follows the same procedure as before where in particular $x > 0$ requires $F(k^*) > q_b^* + k^*$. For $k^* = (1-\sigma)\,q_b^*$, it holds that $k = k^* = (1-\sigma)q_b^* = (1-\sigma)u'(\cdot)q_b$ for all $\pi > \beta$, and so the equilibrium must be just binding credit constraint with $i > 0$ where the same procedure applies to establish its existence and uniqueness. ■

<u>**Case 2-B:** $\alpha = 1$ **and** $k^* < (1-\sigma)q_b^*$**.</u>** An equilibrium is with binding credit-constraint, exists, is unique and satisfies $i = 0$ for $\pi \in (\beta, \underline{\pi}')$, and $i > 0$ for $\pi \in [\underline{\pi}', \infty)$.

Proof of Case 2-B. When $k^* < (1-\sigma)\, q_b^*$ and $\alpha = 1$, an equilibrium must be with binding credit-constraint for any $\pi \in (\beta, \infty)$. Further, as in Case 1-B, there exists a unique cutoff value $\underline{\pi}' \in (\beta, \infty)$ such that equilibrium implies and $i = 0$ for $\pi < \underline{\pi}'$, and $i > 0$ for $\pi \geq \underline{\pi}'$. That is, $q_b \in [\underline{q}_b', q_b^*)$, $k \in (k^*, \underline{k}']$, $\mu \in [0, \mu^*)$ are uniquely determined by (36)-(38) for $\pi \in (\beta, \underline{\pi}']$, while $q_b \in (0, \underline{q}_b')$, $k = \underline{k}'$, $i \in (0, \infty)$ by (39)-(41) for $\pi \in (\underline{\pi}', \infty)$, where $\pi = \underline{\pi}'$ yields $\underline{k}' = (1-\sigma)\underline{q}_b'$ and $i = 0$. Finally, for all $\pi \in (\beta, \infty)$, we have $k^* < k < (1-\sigma)q_b^*$ and so $x > 0$ given $F(k^*) > q_b^* + (1-\sigma)q_b^*$. ∎

<u>**Case 3-A:** $\alpha > 1$ **and** $k^* > (1-\sigma)q_b^*$**.</u>** For $\pi \in (\beta, \tilde{\pi}')$ an equilibrium exists, is unique and implies that the credit constraint is not binding for $\pi \in (\beta, \hat{\pi}')$; the credit constraint is binding with $i > 0$ for $\pi \in [\hat{\pi}', \tilde{\pi}')$, given $\lim_{q_b \to 0} u'(q_b)q_b > k^*/(1-\sigma)$.

Proof of Case 3-A. Observe first from (28) and (29) that $q_b \to 0$, $i \to \infty$ as $\pi \to \infty$. Assuming $\lim_{q_b \to 0} u'(q_b)q_b > k^*/(1-\sigma)$ implies that if an equilibrium exists, then it must be with binding credit-constraint for a sufficiently large π. Hence, given $k = k^* > (1-\sigma)q_b^* = (1-\sigma)u'(q_b)q_b$ as $\pi \to \beta$ and $u'(q_b)q_b$ is strictly increasing in π (when the constraint is not binding and $\alpha > 1$), there exists a unique cutoff value $\hat{\pi}' \in (\beta, \infty)$ that solves (35). That is, an equilibrium is without binding credit-constraint for $\pi < \hat{\pi}'$ and is with binding credit-constraint for $\pi \geq \hat{\pi}'$. As shown in the proof of Case 1-A, $\mu = 0$ and $q_b, i, k > 0$ are a unique solution to (32)-(34) when the constraint is not binding, and so for $\pi \in (\beta, \hat{\pi}')$ an equilibrium exists and is unique given $F(k^*) > k^* + q_b^*$.

For $\pi \geq \hat{\pi}'$, notice q_b must satisfy (39) because $\mu = 0$. Observe that for $\pi \in [\hat{\pi}', \infty)$, $\Phi(\cdot)$ satisfies: $\Phi(\hat{q}_b';\ \pi) > 0$ where $\hat{q}_b' \in (0, q_b^*)$ is given by $u'(\hat{q}_b') = \pi/\beta$; $\Phi(\bar{q}_b',\ \pi) < 0$ where $\bar{q}_b' \in (\beta, \hat{q}_b')$ is given by $u'(\bar{q}_b') = \pi/\sigma\beta$. Therefore, because $\Phi(\cdot)$ is continuous in q_b and $\partial\Phi(\cdot)/\partial q_b > 0$, there exists a unique solution $q_b \in (\bar{q}_b', \hat{q}_b')$ that satisfies $\Phi(\cdot) = 0$ for $\pi \in [\hat{\pi}, \infty)$. Given this solution, k, $i > 0$ are uniquely determined by (40) and (41), respectively.

To guarantee $x > 0$ requires an extra care in this case, since $k \geq k^* > (1-\sigma)q_b^*$ for $\pi \in [\hat{\pi}', \infty)$ (see the proof of Proposition 4). Note, however, that at $\pi = \hat{\pi}'$, we have $q_b = \hat{q}_b' < q_b^*$ and $k = k^* = (1-\sigma)u'(\hat{q}_b')\hat{q}_b'$, and that we can scale $q_b^* = u'^{-1}(1)$ or $k^* = F'^{-1}(1/\beta)$ so that $q_b^* > k^* + \sigma\hat{q}_b$, which implies

$$x = F(k^*) - k^* - \sigma\hat{q}_b' - k^* > F(k^*) - k^* - q_b^* > 0,$$

at $\pi = \hat{\pi}'$ given $F(k^*) > k^* + q_b^*$. Hence, there exists some $\tilde{\pi}' \in (\hat{\pi}', \infty)$ such that $x > 0$ and hence the existence and uniqueness of the equilibrium are guaranteed for $\pi \in [\hat{\pi}', \tilde{\pi}')$. ∎

<u>**Case 3-B:** $\alpha > 1$ **and** $k^* \leq (1-\sigma)q_b^*$**.</u>** If $k^* = (1-\sigma)q_b^*$, an equilibrium exists and unique, with the binding credit-constraint and $i > 0$ for $\pi \in (\beta, \tilde{\pi}'')$. If $k^* < (1-\sigma)q_b^*$, an equilibrium exists and unique, with the binding credit-constraint and $i = 0$ for $\pi \in (\beta, \underline{\pi}'')$, and with the binding credit-constraint and $i > 0$ for $\pi \in [\underline{\pi}'', \tilde{\pi}'')$.

Proof of Case 3-B. Note first that given $\alpha > 1$ and $k^* \leq (1-\sigma)q_b^*$ an equilibrium, if it exists, must be with binding credit-constraint. Observe that when $\pi \to \beta$, (28)-(31) imply $q \to q^*$, $k \to k^*$, $i \to 0$, $\mu \to \mu^* \equiv 1 - \sigma k^*/(1-\sigma)(q_b^* - k^*) \geq 0$ (with equality when $k^* = (1-\sigma)q_b^*$). Consider first the case $k^* < (1-\sigma)q_b^*$. If an equilibrium exists for π close to β, then it must satisfy $i = 0$ and q_b, k, μ are given by equations (36)-(38). As shown in the proof of Case 1-B, a solution $q_b \in [\underline{q}_b'', q_b^*)$, $k \in (k^*, \underline{k}'']$, $\mu \in [0, \mu^*)$ to (36)-(38) exists and is unique for $\pi \in (\beta, \underline{\pi}'')$, where $\pi = \underline{\pi}'' \in (\beta, \infty)$ yields $\underline{k}'' = (1-\sigma)\underline{q}_b''$ (which leads to $\mu = 0$). For $\pi \in [\underline{\pi}'', \infty)$, we must have $\mu = 0$, and $q_b \in (0, \underline{q}_b'')$, $k \in (\underline{k}'', \infty)$, $i \in (0, \infty)$ are unique solution to (39)-(41), where $\pi = \underline{\pi}''$ yields $\underline{k}'' = (1-\sigma)\underline{q}_b''$ and $i = 0$. When $k^* = (1-\sigma)q_b^*$, because $\mu \to 0$ as $\pi \to \beta$, if an equilibrium exists for π close to β, then it must be with binding credit-constraint and $i > 0$. In this case, there exist a unique solution $q_b \in (0, q_b^*)$, $k \in (k^*, \infty)$, $i \in (0, \infty)$ to equations (39)-(41).

Finally, $x > 0$ can be guaranteed for $\pi \in (\beta, \tilde{\pi}'']$ (when $k^* < (1-\sigma)q_b^*$) and for $\pi \in (\beta, \widetilde{\pi}'']$ (when $k^* = (1-\sigma)q_b^*$) given $F(k^*) > q_b^* + (1-\sigma)q_b^*$, where $\pi = \tilde{\pi}''$ (when $k^* < (1-\sigma)q_b^*$) or $\pi = \widetilde{\pi}''$ (when $k^* = (1-\sigma)q_b^*$) yields $k = (1-\sigma)q_b^*$. ■

4.2 Proof of Proposition 4

When the credit constraint is not binding, (34) determines $k = k^*$ which is independent of π. When the credit constraint is binding with $i > 0$, (37) determines $k = k(\pi) > k^*$ which is strictly increasing in $\pi > \beta$. When the credit constraint is binding with $i = 0$, (40) determines $k = k(\pi) \geq k_*$, given $q_b > 0$ satisfies (39). In this case, noting that

$$\frac{dq_b}{d\pi} = \frac{-(\frac{\pi}{\beta} - \sigma u'(\cdot))^2 F''(\cdot) q_b + \sigma(1-\sigma)u'(\cdot)}{\sigma(1-\sigma)\pi u''(\cdot) + \beta\left(\frac{\pi}{\beta} - \sigma u'(\cdot)\right)^2 F''(\cdot)\left(\frac{\pi}{\beta} - \sigma(u'(\cdot) + u''(\cdot)q_b)\right)} < 0, \tag{42}$$

gives:

$$\begin{aligned}\frac{dk}{d\pi} &= \frac{q_b}{\beta} + \left(\frac{k}{q_b} - \sigma u''(\cdot)q_b\right)\frac{dq_b}{d\pi} \\ &= \frac{\sigma(1-\sigma)\frac{k}{q_b}u'(\cdot)(1-\alpha)}{\sigma(1-\sigma)\pi u''(\cdot) + \beta\left(\frac{k}{q_b}\right)^2 F''(\cdot)\left(\frac{k}{q_b} - \sigma u''(\cdot)q_b\right)} \gtreqless 0 \quad \text{if and only if} \quad \alpha \gtreqless 1.\end{aligned}$$

■

4.3 Proof of Corollary 1

There are two cases for $\mu = 0$. First, when the credit constraint is not binding, (32) determines $q_b = q_b(\pi)$ and implies that:

$$\alpha_\pi \equiv -\frac{dq_b/d\pi}{q_b/\pi} = -\frac{\pi/q_b}{\beta u''(\cdot)} = -\frac{u'(\cdot)}{u''(\cdot)q_b} \equiv \frac{1}{\alpha}$$

which yields: $\alpha_\pi \gtreqless 1$ if and only if $\alpha \lesseqgtr 1$. Second, when the credit constraint is binding, (39) determines $q_b = q_b(\pi)$. In this case, applying (39) to (42) yields:

$$\alpha_\pi = \frac{\left(\frac{k}{q_b}\right)^2 F''(\cdot)\pi - \sigma(1-\sigma)u'(\cdot)\frac{\pi}{q_b}}{\sigma(1-\sigma)\pi u''(\cdot) + \beta\left(\frac{k}{q_b}\right)^2 F''(\cdot)\left(\frac{\pi}{\beta} - \sigma(u'(\cdot) + u''(\cdot)q_b)\right)} \gtreqless 1$$

$$\begin{aligned}&\iff \sigma(1-\sigma)\pi\frac{u'(\cdot)}{q_b}\left(1 + \frac{u''(\cdot)q_b}{u'(\cdot)}\right) \gtreqless \left(\frac{k}{q_b}\right)^2 F''(\cdot)\beta\sigma u'(\cdot)\left(1 + \frac{u''(\cdot)q_b}{u'(\cdot)}\right) \\ &\iff \left((1-\sigma)\frac{\pi}{q_b} - \left(\frac{k}{q_b}\right)^2 F''(\cdot)\beta\right)\sigma u'(\cdot)(1-\alpha) \gtreqless 0.\end{aligned}$$

In the last expression above, observe that the terms in the bracket are positive, thereby we have: $\alpha_\pi \gtreqless 1$ if and only if $\alpha \lesseqgtr 1$. ■

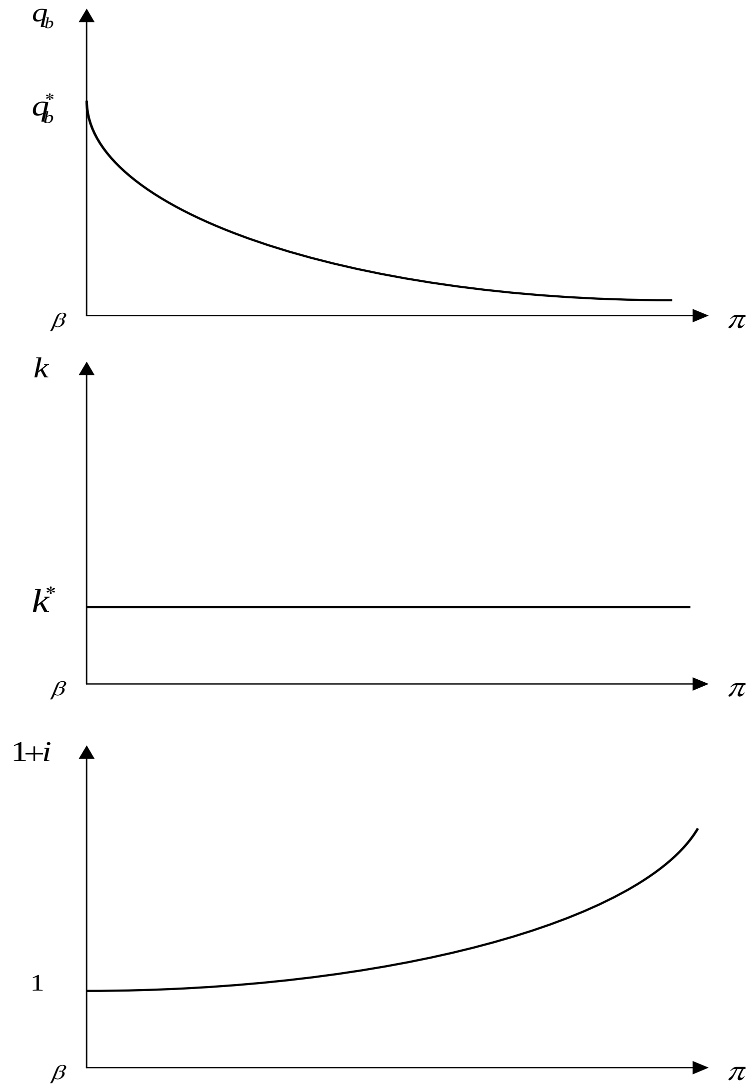

Figure 1: Steady state equilibrium with $\alpha \leq 1$ and $k^* \geq (1-\sigma)q_b^*$

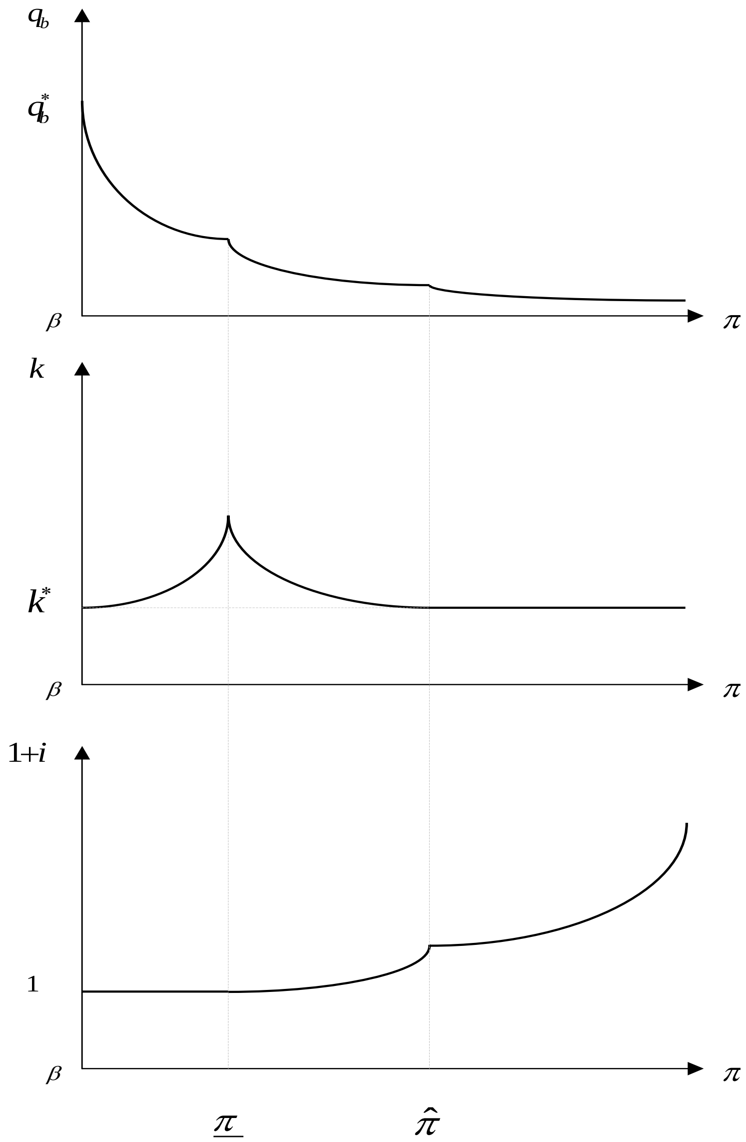

Figure 2: Steady state equilibrium with $\alpha < 1$ and $k^* < (1-\sigma)q_b^*$

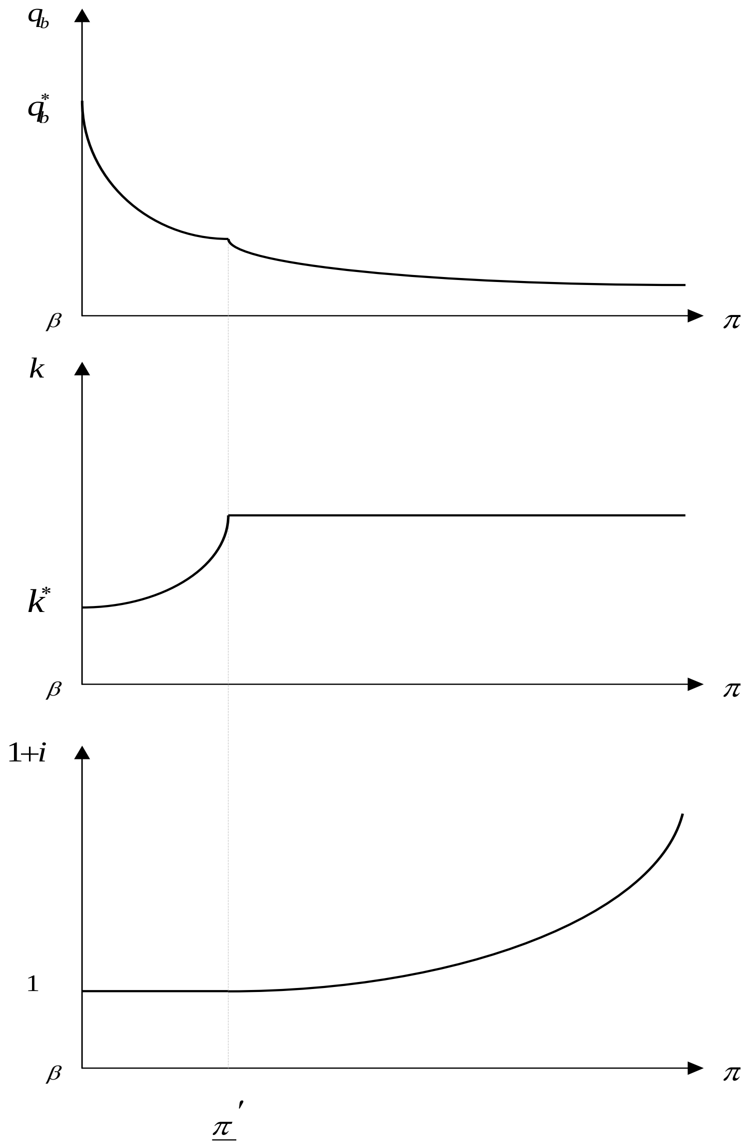

Figure 3: Steady state equilibrium with $\alpha = 1$ and $k^* < (1-\sigma)q_b^*$

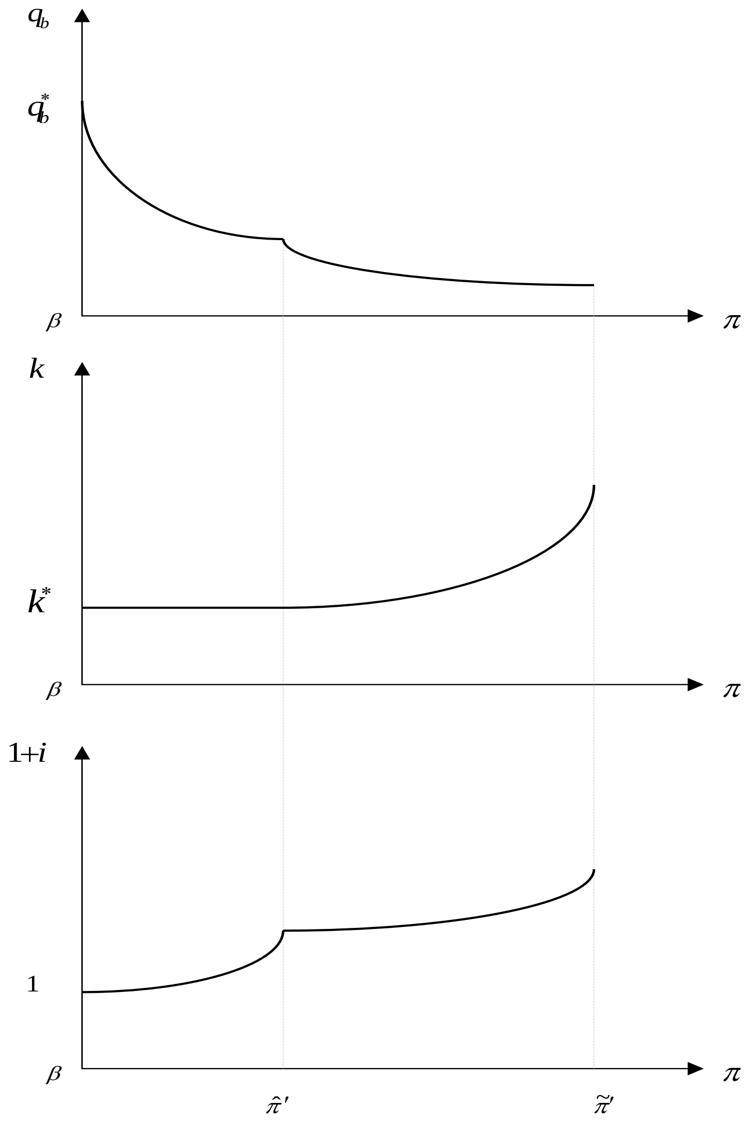

Figure 4: Steady state equilibrium with $\alpha > 1$ and $k^* > (1-\sigma)q_b^*$

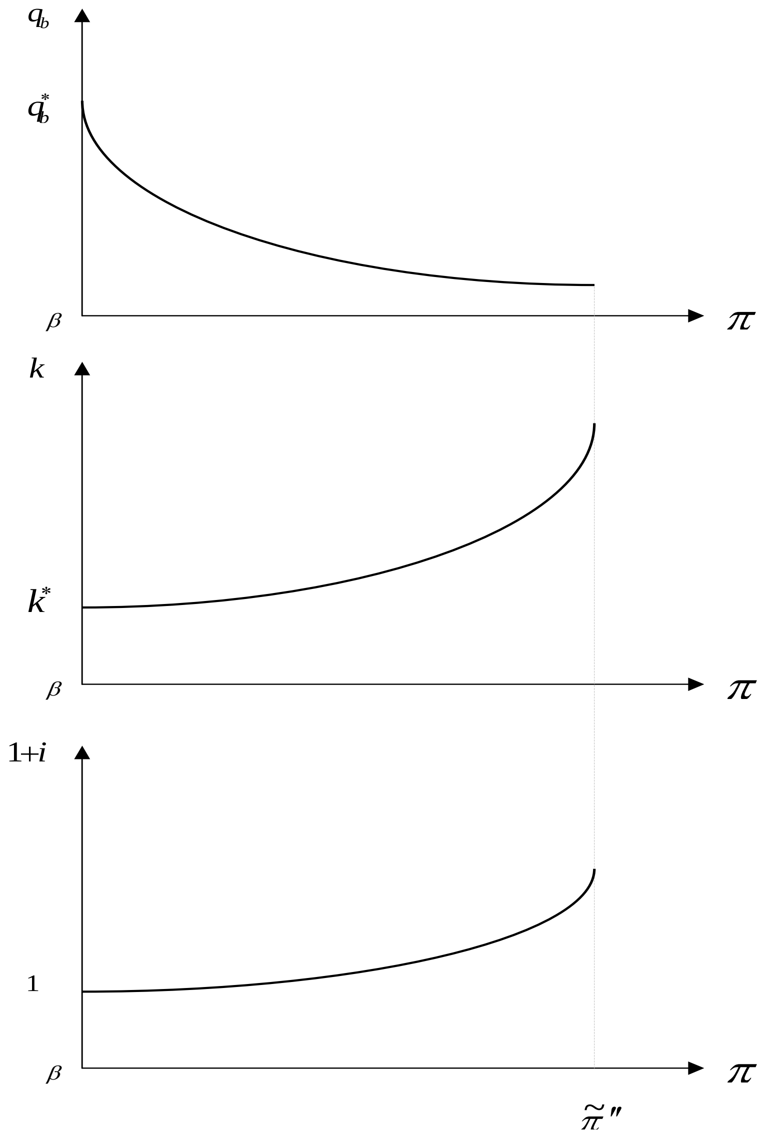

Figure 5: Steady state equilibrium with $\alpha > 1$ and $k^* = (1-\sigma)q_b^*$

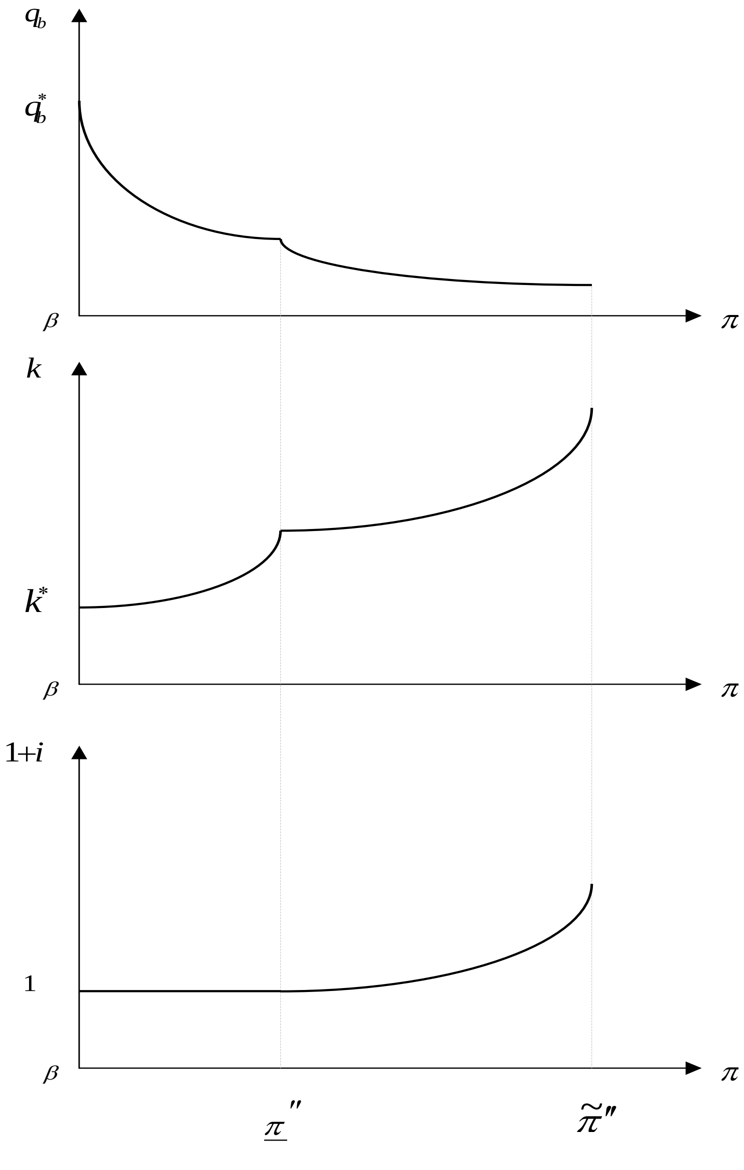

Figure 6: Steady state equilibrium with $\alpha > 1$ and $k^* < (1-\sigma)q_b^*$