The Societal Benefits of Outside Versus Inside Bonds*

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Abstract

When agents are cash constrained, two options exist – borrow or sell assets. We compare the welfare properties of these options in two economies: in one, agents can borrow (issue inside bonds) and in the other they can sell 'illiquid' government bonds (outside bonds). All transactions are voluntary, implying no taxation or forced redemption of private debt. We show that any allocation in the economy with inside bonds can be replicated in the economy with outside bonds. Moreover, under the best policies, the allocation with outside bonds strictly dominates the allocation with inside bonds.

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1 Introduction

In monetary models, agents often face binding cash constraints. Two solutions have been proposed for improving on the monetary allocation. Kocherlakota (2003) shows that the presence of 'illiquid' nominal government bonds (*outside* bonds) allows those who are in need of cash to sell bonds which improves the allocation. Using a different monetary model, Berentsen, Camera and Waller (2006) show that credit (or *inside* bonds) improves the allocation since it allows agents to borrow or lend cash depending on their liquidity needs. These results raise the following three questions: First, within a common monetary framework, when are the allocations with illiquid outside bonds and inside bonds the same? Second, when do they differ and why? Third, what is the optimal monetary policy in each case? Our focus in this paper is to address these questions.[1]

We construct a general equilibrium monetary model in the spirit of Lagos and Wright (2005) with fiat money and one-period bonds. Within this framework, we consider two economies, one in which agents hold government bonds and one in which they hold inside bonds. Each period consumers receive idiosyncratic preference shocks which creates differential needs for liquidity across consumers. After these shocks occur, but before the goods market opens, agents can trade bonds for money. Those with high liquidity needs sell bonds while those with low liquidity acquire bonds. As in Kocherlakota (2003), we assume that the outside bonds are illiquid in the sense that they cannot be used as a medium of exchange in the goods market. More importantly, we assume all trades must be voluntary. This implies that the government cannot imposes taxes or run a deflation since it requires lump-sum taxation of money balances. It also implies that, in the inside bond economy, redemption of inside bonds must be voluntary. We study the effects of steady state inflation on the allocations as opposed to one-time changes in the stock of outside bonds as done by Kocherlakota.

We show that for any positive inflation rate, bonds are essential in both economies, and thus generate societal benefits.[2] With regards to the first and second questions, we find that for sufficiently high inflation rates, the allocations in the two economies are the same but they differ for low inflation rates. The key result that emerges from our paper is with regard to the third question: under optimal policies, the optimal allocation with illiquid outside bonds dominates the optimal

[1]We would like to thank Neil Wallace for suggesting we pursue this line of research.

[2]By essential we mean that the use of money expands the set of allocations (see Kocherlakota (1998) and Wallace (2001)).

allocation with inside bonds. To prove this result, we first show that any allocation attained in the economy with inside bonds can be replicated in the economy with outside bonds and that the converse is not true. In particular, the best allocation with outside bonds cannot be replicated with inside bonds. The crucial friction that drives this result is lack of enforcement. To verify this, we also show that if redemption of loans can be forced in the economy with inside bonds, then the best allocation with ouside bonds can be replicated with inside bonds.

Our results are in contrast to Wallace's (1981) famous Modigliani-Miller result for open market operations which is that the money/bond composition of a government's debt portfolio does not affect the equilibrium allocation. They also differ from the results of a recent paper by Kocherlakota (2007). He considers various models of asset trade. In these models agents can either trade a privately issued one-period bond, a publicly issued one-period bond, or a publicly issued money. He proves that there are enforcement policies such that the allocations are equivalent.[3] As noted by Kocherlakota the crucial assumptions for these results are that the government has access to lump-sum taxes, which require collection power, and absence of a transaction advantage of money over bonds. In contrast, in our model the money/bond composition affects the allocation and the allocations in the economies with inside or outside bonds can differ. The reason is that in our economies money provides transaction services and we neither allow for public enforcement, i.e., no lump-sum taxes, nor private enforcement of credit contracts. In his paper, Kocherlakota emphasizes that many results in the literature rely on asymmetric collection powers of private and government entities. To eliminate this asymmetry he assumes equal and positive collection powers of private agents and the government. We eliminate it by assuming neither has any collection power.

Several papers are related to what we do here. Kehoe and Levine (2001) compare allocations in a dynamic economy when agents can acquire consumption goods in one case by selling their capital holdings and in another case by issuing debt subject to a borrowing constraint. They show that if agents are sufficiently patient, the allocations are the same in a deterministic environment but if they are sufficiently impatient, then the debt constrained allocation leads to a better allocation. Shi (2006) examines the implications of illiquid bonds in a monetary search model where there are legal restrictions preventing bonds from being used as a medium of exchange in some transactions but not others. The legal restrictions make outside bonds illiquid relative to money. He finds that

[3]In an earlier paper Taub (1994) derived a related equivalence result between money and credit.

having illiquid bonds can be welfare improving. In Boel and Camera (2006) bonds are illiquid in the sense that there is a transaction fee for converting them into cash. Since agents have different discount factors and trading opportunities, for some parameter configurations, there is a welfare improving role for illiquid bonds under the optimal monetary policy. Finally, our work is also related to Lagos and Rocheteau (2003) who study the use of illiquid bonds in a variant of the LW model. They find that under the optimal monetary policy (zero inflation) illiquid bonds are inessential.[4]

Finally, the paper is also related to Cavalcanti and Wallace (1999a, 1999b) who study the relation between inside and outside money. Their model is related to ours since some buyers are able to relax their cash constraint by issuing personal liabilities to sellers which improves the allocation. However, the inefficiency associated with holding idle cash balances is not eliminated. In our model agents can lend their idle cash balances and earn interest. In particular, we find that the welfare gain associated with inside bonds is not due to relaxing buyers' cash constraints rather it arises from generating a positive rate of return earned on idle cash balances.[5]

The structure of the paper is as follows. In Section 2 we describe the environment. Section 3 contains analysis of the economy with outside bonds. Section 4 examines the economy with inside bonds and Section 5 compares the allocation of the two economies. Section 6 concludes.

2 The environment

The environment builds on Lagos and Wright (2005) and Berentsen, Camera and Waller (2006).[6] Time is discrete and in each period three perfectly competitive markets open and close sequentially.[7] There is a [0,1] continuum of infinitively-lived agents.

The timing of the model is as follows. The first market is a financial market where agents trade money for bonds. The second market is a goods market where agents trade money for market 2

[4] Furthermore, there are a number of papers that study the coexistence of money and bonds (i.e. Diaz-Perrera-Tallo (2007), Ferris and Watanabe (2007), Telyukova and Wright (2007), Marchesiani and Senesi (2007). The key difference to our work is that they never compare the allocative effects of different bonds.

[5] This shows that being constrained is not per se a source of inefficiency. In any general equilibrium model, agents face binding budget constraints. Nevertheless, the equilibrium is efficient if all gains from trade are exploited.

[6] An alternative framework would be Shi (1997) which we could amend with preference and shocks and bond markets to generate the same results.

[7] Competitive pricing in the Lagos-Wright framework has been introduced by Rocheteau and Wright (2004) and further investigated in Berentsen, Camera and Waller (2005) and Lagos and Rocheteau (2005).

goods. In the third market all agents can produce and consume goods. At the beginning of the first market, agents get a preference shock that determines whether they can produce or consume in the second market. With probability $1-n$ an agent can consume and cannot produce. We refer to these agents as buyers. With probability n, an agent can produce and cannot consume. These are sellers.

Moreover, buyers receive an idiosyncratic preference shock to utility. They learn that they will get utility $\varepsilon u(q)$ from q consumption in the second market, where $u'(q), -u''(q) > 0$. The shock ε has a continuous distribution $F(\varepsilon)$ with support $[0, \varepsilon_H]$, is serially uncorrelated and has expected value $\bar{\varepsilon} = \int_0^{\varepsilon_H} \varepsilon dF(\varepsilon)$.[8] Producers in the second market incur a utility cost $c(q) = q$ from producing q units of output. All trades in market 2 are anonymous and agents' trading histories in this market are private information, thus no trade credit exists. Hence there is a role for money, as sellers require immediate compensation for their production effort.

Following Lagos and Wright (2005) we assume that agents receive utility $U(x)$ from x consumption, with $U'(x), -U''(x) > 0$, $U'(0) = \infty$, and $U'(+\infty) = 0$. They can also produce these goods with a constant returns to scale production technology where one unit of the consumption good is produced with one unit of labor h generating one unit of disutility.[9] The discount factor across periods is β.

2.1 First-best allocation

The expected steady state lifetime utility of the representative agent at the beginning of the period before types are realized is

$$(1-\beta)\mathcal{W} = \int_0^{\varepsilon_H} \left[(1-n)\varepsilon u(q_\varepsilon) - nq_s\right] dF(\varepsilon) + U(x) - h. \tag{1}$$

where q_ε is consumption and q_s production in market 2. We use (1) as our welfare criteria.

To derive the welfare maximizing quantities we assume that all agents are treated symmetrically.

[8] All of our results go through with a non-zero lower bound. Setting the lower bound of ε to zero simplifies the presentation of the results.

[9] As in Lagos and Wright (2005), these assumptions allow us to get a degenerate distribution of money holdings at the beginning of a period. The different utility functions $U(.)$ and $u(.)$ allow us to impose technical conditions such that in equilibrium all agents produce and consume in the last market.

The planner then maximizes (1) subject to the feasibility constraint

$$Q \equiv (1-n) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) = nq_s. \tag{2}$$

where Q is aggregate consumption. The first-best allocation satisfies

$$\begin{aligned} U'(x^*) &= 1 \text{ and} \\ \varepsilon u'[q^*(\varepsilon)] &= 1 \text{ for all } \varepsilon. \end{aligned} \tag{3}$$

These are the quantities chosen by a social planner who could force agents to produce and consume.

2.2 Outside bonds versus inside bonds

We analyze equilibria of the model under two different bond markets – a market for outside bonds and one for inside bonds. We do so in order to see if one of the bond markets generates a superior allocation relative to the other. Outside bonds are nominal government debt obligations whereas inside bonds are private debt obligations. We assume that the government has a record-keeping technology over bond trades and acts as the intermediary in the bond market. Consequently, bond holdings can simply be book-keeping entries – no physical object exists. This makes bonds incapable of being used as media of exchange hence they are illiquid. However, the government has no record-keeping technology over goods exchange and a buyer's promise to deliver outside bonds to a seller in the next market is not credible. Hence, fiat money is essential for trade in market 2. Because of this structure, agents must hold non-negative quantities of outside bonds.

Inside bonds are financial claims on private agents issued in a private bond market. Consequently, issuing inside bonds is equivalent to receiving credit as in Berentsen, Camera and Waller (2006). We assume a perfectly competitive financial market exists where intermediaries have a record-keeping technology over financial trades. The intermediaries acquire nominal debt obligations from agents and issue nominal debt obligations on themselves, which are securitized by their acquired claims on private agents. In this sense, private agents are still anonymous to each other but not to the financial intermediary.[10] However, this record-keeping technology does not exist in

[10] An example is a bank who accepts nominal deposits and makes nominal loans. While the bank knows who it trades with, borrowers do not know the identity of depositors and vice versa.

the goods market thus ruling out trade credit between buyers and sellers of goods.

In any model of credit, default is a serious issue. We consider an environment where repayment is voluntary – creditors have no powers to collect unpaid debts. The only punishment for default is exclusion from the financial sector in all future periods. Given this punishment, we derive conditions to ensure voluntary redemption and show that this may involve binding borrowing constraints, i.e. credit rationing. We assume the financial intermediaries honor their debt obligations.

2.3 Government

In the model with outside bonds, we assume a government exists that controls the supply of fiat currency and issues one-period nominal bonds. Government bonds are perfectly divisible, payable to the bearer and default free. One bond pays off one unit of currency at maturity. Denote M_t the per capita money stock and B_t the per capita stock of newly issued bonds in market 3 in period t. The change in the money stock is given by

$$M_t - M_{t-1} = \tau_t M_{t-1} + B_{t-1} - \rho_t B_t + P_t G_t \tag{4}$$

where $P_t G_t$ is the nominal amount of government spending in period t in the centralized market and P_t is the money price of goods in market 3. The total change in the money stock is comprised of three components: first, a lump-sum transfer of cash (a 'gift' of cash); second, the net difference between the cash created to redeem bonds, B_{t-1}, and the net cash withdrawal from selling B_t units of bonds at the price ρ_t; and finally, the cash printed to pay for government goods. We assume there are positive initial stocks of money and outside bonds M_0 and B_0. For $\tau_t < 0$, the government must be able to extract money via lump-sum taxes from the economy. Throughout the paper we assume that $\tau_t < 0$ is not feasible because, as for the private sector, the government has no collection power.[11] We can then write this expression as

$$M_t = \gamma_t M_{t-1} \tag{5}$$

[11] The inability to impose lump-sum taxes occurs in environments with limited enforcement. In such environments all trades must be voluntary and so lump-sum taxes of money are not feasible because the central bank cannot impose any penalties on the agents. If she could impose such penalties there would be no role for money since "producers could be forced to produce for households" (Kocherlakota 2003, p. 185). This implies that the government cannot run the Friedman rule which would implement the first-best allocation in this environment.

where γ_t is the gross growth rate of the money supply in period t.

To simplify the analysis, we assume $G_t = 0$ for all t. This implies that all money creation comes from paying off net nominal bond obligations $B_{t-1} - \rho_t B_t$ and the lump-sum gifts of money $\tau_t M_{t-1}$. For the case of outside bonds we assume that $\tau_t = 0$. Consequently, 4) reduces to[12]

$$M_t - M_{t-1} = B_{t-1} - \rho_t B_t. \tag{6}$$

This allows us to focus solely on how 'open market operations' affect the equilibrium allocation and allows us to ignore optimal taxation issues to finance government spending. Using (4) and (5) yields

$$(\gamma_t - 1)\,\frac{M_{t-1}}{B_{t-1}} = (1 - \rho_t \eta_t) \tag{7}$$

where $\eta_t \equiv B_t / B_{t-1}$ is the gross growth rate of bonds. This equation relates the gross growth rate of money γ_t to the gross growth rate of bonds η_t. For a given ratio of bonds to money and a given discount on bonds an increase in η_t requires a decrease in γ_t.

In the model with inside bonds, we assume that $B_t = 0$ in all periods but the government still controls the amount of fiat currency in the economy. In this case, agents receive lump-sum gifts of money $\tau_t M_{t-1} \geq 0$ and the money supply grows according to $M_t = (1 + \tau_t)\, M_{t-1} = \gamma_t M_{t-1}$.

2.4 Stationary equilibria

In period t, let $\phi_t = 1/P_t$ be the real price of money in market 3. For notational ease, variables corresponding to the next period are indexed by $+1$, and variables corresponding to the previous period are indexed by -1. We focus on symmetric and stationary equilibria where all agents follow identical strategies and where real allocations are constant over time. In a stationary equilibrium end-of-period real money balances are time-invariant

$$\phi M = \phi_{+1} M_{+1}. \tag{8}$$

Moreover, we restrict our attention to equilibria where γ is time invariant which implies that $\phi/\phi_{+1} = P_{+1}/P = M_{+1}/M = \gamma$.

[12] All our results continue to hold for $\tau_t > 0$. The case $\tau_t < 0$ is not feasible. See the previous footnote.

3 Outside bonds

In this Section we analyze the economy with outside bonds. Let $V_1(m,b)$ denote the expected value from entering market 1 with m units of money and b outside bonds, $V_{2j}(m,b)$, $j=\varepsilon,s$, the expected value from entering market 2 with m units of money and b bonds, and $V_3(m,b)$ the expected value from entering market 3 with m and b. For notational simplicity we suppress the dependence of the value function on the time index t. In what follows we look at a representative period t and work backward, from the third to the first market.

The third market In the third market, the problem of a representative agent is:

$$V_3(m,b) = \max_{x,h,m_{+1},b_{+1}} U(x) - h + \beta V_1(m_{+1},b_{+1})$$
$$s.t. \quad x + \phi m_{+1} + \phi\rho b_{+1} = h + \phi m + \phi b.$$

where ρ is the money price of bonds in the third market. Using the budget constraint to eliminate h in the objective function, one obtains the first-order conditions $U'(x)=1$ and

$$\beta V_1^m(m_{+1},b_{+1}) \leq \phi \;(=0 \text{ if } m_{+1}>0\,) \tag{9}$$
$$\beta V_1^b(m_{+1},b_{+1}) \leq \phi\rho \;(=0 \text{ if } b_{+1}>0\,). \tag{10}$$

$V_1^m(m_{+1},b_{+1})$ is the marginal value of taking an additional unit of money into period $t+1$. Since the marginal disutility of working is 1, $-\phi$ is the utility cost of acquiring one unit of money in the third market of period t. $V_1^b(m_{+1},b_{+1})$ is the marginal value of taking additional bonds into period $t+1$. Here, $-\phi\rho$ is the utility cost of acquiring a bond in the third market of period t. The implication of (9) and (10) is that all agents enter the following period with the same amounts of money and bonds.

The envelope conditions are

$$V_3^m = \phi; V_3^b = \phi \tag{11}$$

As in Lagos-Wright (2005) the value function is linear in wealth.

The second market Let q_ε denote the quantities consumed by a type ε buyer and q_s the quantity produced by a seller trading in market 2. Let p be the nominal price of goods in market 2.

A seller who holds m money and b bonds at the opening of the second market has expected lifetime utility $V_{2s}(m,b) = \max_{q_s}\left[-q_s + V_3(m + pq_s, b)\right]$. Using (11), the first-order condition reduces to

$$p = 1/\phi. \tag{12}$$

Consequently, sellers are indifferent on how much they produce. Nevertheless, in a symmetric equilibrium they all produce the same amount.

A type ε buyer has expected lifetime utility $V_{2\varepsilon}(m,b) = \max_{q_\varepsilon}\left[\varepsilon u(q_\varepsilon) + V_3(m - q_\varepsilon/\phi, b)\right]$ s.t. $q_\varepsilon \leq \phi m$ where (12) has been used to eliminate p. Using (11) and (12) the buyer's first-order conditions can be written as

$$\varepsilon u'(q_\varepsilon) = 1 + \lambda_\varepsilon \tag{13}$$

where λ_ε is the multiplier of the buyer's budget constraint. If the budget constraint is not binding, $\lambda_\varepsilon = 0$. If it is binding, then $\varepsilon u'(q_\varepsilon) > 1$ which means trades are inefficient. In this case the buyer spends all of his money.

Using the envelope theorem, the marginal values of bonds and the marginal values of money for buyers and sellers at the beginning of the second market are

$$V_{2\varepsilon}^b = V_{2s}^b = \phi \tag{14}$$

$$V_{2\varepsilon}^m = \phi\varepsilon u'(q_\varepsilon) \text{ and } V_{2s}^m = \phi. \tag{15}$$

Finally, market clearing satisfies (2).

The first market Let y_j denote the quantity of outside bonds *bought* by an agent of type $j = \varepsilon, s$ in market 2. Let a be the price of bonds in market 1, consequently the nominal interest rate earned by acquiring a bond in this market is $i = (1-a)/a$, which is greater than zero if and only if $a < 1$. Note that there are two short-selling constraints: agents cannot sell more bonds or money than they hold.

An agent who holds m money and b bonds at the opening of the first market has expected lifetime utility

$$V_1(m,b) = (1-n)\int_0^{\varepsilon_H} V_{2\varepsilon}\left[m - y_\varepsilon/(1+i), b + y_\varepsilon\right] dF(\varepsilon) + nV_{2s}\left[m - y_s/(1+i), b + y_s\right] \tag{16}$$

where, for $j = \varepsilon, s$, $y_j = \arg\max_y V_{2j}\left[m - y/(1+i), b + y\right]$ s.t. $m - y/(1+i) \geq 0$ and $b + y \geq 0$. The first-order condition is

$$-V_{2j}^m/(1+i) + V_{2j}^b - \phi\mu_j/(1+i) + \phi\theta_j = 0 \tag{17}$$

where $\phi\mu_j$ is the Lagrange multiplier on the short-selling constraint $m - y_j/(1+i) \geq 0$ and $\phi\theta_j$ is the Lagrange multiplier on $b + y_j \geq 0$. Obviously, both can not bind at the same time.

Consider first an agent who will be a producer in market 2. If $i < 0$, then $y_s = -b$ and goods producers sell their bonds for money in market 1. This obviously cannot be an equilibrium and is ignored for the remainder of the paper which means that $\theta_s = 0$. We can then use (14) and (15) to substitute V_{2j}^m and V_{2j}^b in (17) to get

$$\mu_s = i \tag{18}$$

If $i > 0$, it is optimal to sell the entire money holdings for bonds. If $i = 0$, the producer is indifferent on how much money to supply. Thus, a producer's bond demand is

$$\begin{array}{lll} y_s \in [-b, m(1+i)] & \text{if} & i = 0 \\ y_s = m(1+i) & \text{if} & i > 0 \end{array} \tag{19}$$

Consider next an agent who will be a buyer in market 2. Since Inada conditions are assumed on $u(q)$ a buyer will always carry some money from market 1 to market 2. Thus, $\mu_\varepsilon = 0$. Accordingly, we can use (14) and (15) to write (17) as follows

$$\varepsilon u'(q_\varepsilon) = (1 + \theta_\varepsilon)(1+i). \tag{20}$$

If $\varepsilon u'(q_\varepsilon) = 1 + i$, then $\theta_\varepsilon = 0$ and $y_\varepsilon \leq b$. In this case the buyer is indifferent between holding bonds or money. In this case, an increase in i makes bonds more attractive relative to holding onto the cash for consumption in market 2. Consequently, buyers trade more of their money balances for illiquid outside bonds, which reduces q_ε.

Finally, if $\varepsilon u'(q_\varepsilon) > 1+i$, then $\theta_\varepsilon > 0$ and he sells all of his bonds implying $y_\varepsilon = -b$. Thus, a buyer's bond demand is

$$\begin{array}{lll} y_\varepsilon \in [-b, m(1+i)] & \text{if} & \varepsilon u'(q_\varepsilon) = 1+i \\ y_\varepsilon = -b & \text{if} & \varepsilon u'(q_\varepsilon) > 1+i. \end{array} \tag{21}$$

Because a buyer's desired consumption is increasing in ε, there is a critical value for the taste index $\tilde{\varepsilon}$ such that

$$\tilde{\varepsilon} u'(\tilde{q}) = 1+i \tag{22}$$

If $\varepsilon \leq \tilde{\varepsilon}$, he does not sell all his bonds while if $\varepsilon \geq \tilde{\varepsilon}$ he sells all his bonds and consumes

$$\tilde{q} = \phi m + \phi b / (1+i). \tag{23}$$

Accordingly, a buyer's consumption satisfies

$$q_\varepsilon = \begin{cases} u'^{-1}[(1+i)/\varepsilon] & \text{if} \quad \varepsilon \leq \tilde{\varepsilon} \\ u'^{-1}[(1+i)/\tilde{\varepsilon}] & \text{if} \quad \varepsilon \geq \tilde{\varepsilon} \end{cases} \tag{24}$$

Finally, applying the envelope theorem to equation (16), and using equations (14) and (15), the marginal values of money and bonds satisfy

$$\frac{\partial V_1(m,b)}{\partial m} = (1-n)\int_0^{\varepsilon_H} \phi\left[\varepsilon u'(q_\varepsilon) + \mu_\varepsilon\right] dF(\varepsilon) + n\phi(1+\mu_s) \tag{25}$$

$$\frac{\partial V_1(m,b)}{\partial b} = (1-n)\int_0^{\varepsilon_H} \phi(1+\theta_\varepsilon) dF(\varepsilon) + n\phi(1+\theta_s) \tag{26}$$

3.1 Equilibrium

To derive the symmetric stationary equilibrium we have to solve for the equilibrium quantities and prices. In any symmetric equilibrium $m = M_{-1}$ and $b = B_{-1}$. Then use (25) and the first-order conditions (9) along with $a = 1/(1+i)$ to get

$$\frac{a\gamma}{\beta} = (1-n)\int_0^{\varepsilon_H} (1+\theta_\varepsilon)dF(\varepsilon) + n \tag{27}$$

and (10) and (26) to get

$$\frac{\rho_{-1}\gamma}{\beta} = (1-n)\int_0^{\varepsilon_H} (1+\theta_\varepsilon)dF(\varepsilon) + n. \tag{28}$$

This implies that in any symmetric stationary equilibrium $\rho_{-1} = a$ and so from now on we set $\rho_{-1} = a$ and ignore the second equation. Using (18), (20) and (27) we obtain

$$\frac{\gamma-\beta}{\beta} = (1-n)\int_0^{\varepsilon_H} \left[\varepsilon u'(q_\varepsilon) - 1\right] dF(\varepsilon) + ni. \tag{29}$$

Using (24) and rearranging yields

$$\frac{\gamma-\beta(1+i)}{\beta(1+i)} = (1-n)\int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1\right) dF(\varepsilon). \tag{30}$$

This is an equation in i and $\tilde{\varepsilon}$. We now derive a second equation i and $\tilde{\varepsilon}$ which we can then use to define the equilibrium.

In any stationary equilibrium the stocks of bonds B and money M must grow at the same rate γ implying a is constant so $a = 1/(1+i) = \rho$. We can then rewrite the government budget constraint (6) to solve for $1+i$ as a function of γ and M_{-1}/B_{-1}

$$1+i = \frac{\gamma}{1-(\gamma-1)\frac{M_{-1}}{B_{-1}}} \tag{31}$$

If $\gamma = 1$, then $i = 0$. In a stationary equilibrium $M_{-1}/B_{-1} = M_0/B_0$ for all t. A non-negative nominal interest rate requires the denominator to be positive or $1 + B_0/M_0 > \gamma$, and $\gamma \geq 1$. Thus, for a given ratio of nominal outside bonds to outside money, the range of feasible γ is bounded by this expression. Thus, define $\bar{\gamma} \equiv 1 + B_0/M_0$ and as $\gamma \to \bar{\gamma}$, $i \to \infty$ and all quantities go to zero.

Substitute (31) into (22) and (30) to get

$$\tilde{\varepsilon}u'(\tilde{q}) = \frac{\gamma}{1-(\gamma-1)\frac{M_0}{B_0}} \tag{32}$$

$$\frac{1-(\gamma-1)M_0/B_0}{\beta}-1 = (1-n)\int_{\tilde{\varepsilon}}^{\varepsilon_H}\left(\frac{\varepsilon}{\tilde{\varepsilon}}-1\right)dF(\varepsilon) \tag{33}$$

For given values of γ and M_0/B_0, the first equation pins down $\tilde{q}$ as a function of $\tilde{\varepsilon}$ while the second determines $\tilde{\varepsilon}$. Thus, if a unique value $0 \leq \tilde{\varepsilon} \leq \varepsilon_H$ solves this equation then we have a unique stationary equilibrium. Given $\tilde{q}$ and $\tilde{\varepsilon}$ we can then solve for all endogenous quantities and prices.

Definition 1 *A stationary monetary equilibrium is an $\tilde{\varepsilon}$ that satisfies (33).*

In what follows define $\gamma_H \equiv 1+(1-\beta)B_0/M_0 < \bar{\gamma}$.

Proposition 1 *For $1 \leq \gamma < \bar{\gamma}$ a unique stationary monetary equilibrium exists. If $\gamma \leq \gamma_H$, then $\tilde{\varepsilon} \in (0, \varepsilon_H]$. If $\gamma_H < \gamma < \bar{\gamma}$ then $\tilde{\varepsilon} > \varepsilon_H$.*

The essence of this proposition is that for sufficiently low inflation rates, high ε buyers will face binding bond sale constraints and so $\varepsilon u'(q_\varepsilon) > 1+i$. Thus, on the margin they would like to sell more bonds but do not have them. For sufficiently high inflation rates, no buyers face binding bond sales constraints and so $\varepsilon u'(q_\varepsilon) = 1+i$ for all ε and from (33) we have $1+(1-\gamma)M_0/B_0 = \beta$ and so from (31)

$$i = \frac{\gamma-\beta}{\beta}. \tag{34}$$

Note that illiquid bonds trade at a discount if and only if $\gamma > 1$. If $\gamma = 1$, then $i = 0$ ($a = 1$).

3.2 Policy implications

We now discuss three policy implications of our model. First, we explore whether the allocation with illiquid bonds differs from the allocation when they are liquid. Second, we would like to know whether inflation can be welfare improving and, third, what is the optimal monetary policy when bonds are illiquid.

Liquid bonds Suppose now that instead of book-keeping entries, bonds are tangible objects that can be used as a medium of exchange in market 2, i.e., they are liquid. One can show that in this environment, the allocation is the same as the allocation without bonds. The intuition and proof for this result is straightforward and provided in Kocherlakota (2003, p. 184): "If bonds are liquid as money, then people will only hold money if nominal interest rates are zero. But then the bonds can just be replaced by money: there is no difference between the two instruments at all." An interesting implication of this result is that "any essentiality of nominal bonds can be traced directly to their (relative) illiquidity (Kocherlakota 2003, p. 184)."

It is straightforward to show that with liquid outside bonds the quantities solve

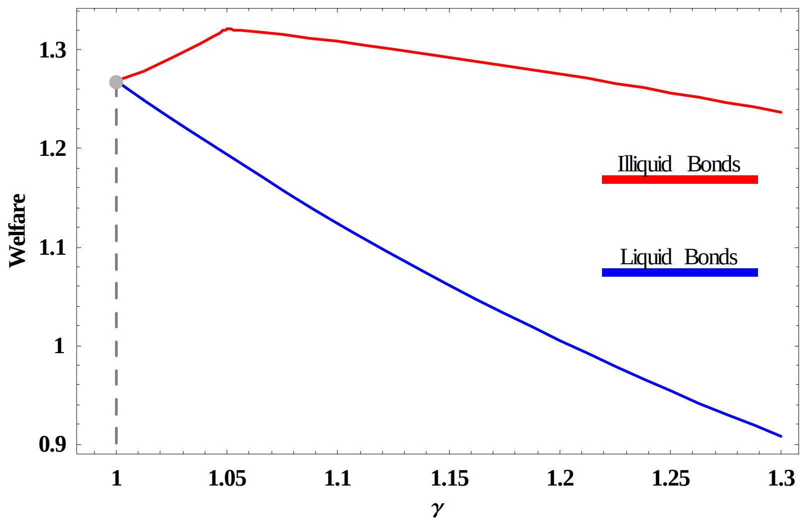
$$\frac{\gamma-\beta}{\beta}=(1-n)\int_{0}^{\varepsilon_H}\left[\varepsilon u'\left(q_{\varepsilon}\right)-1\right]dF\left(\varepsilon\right) \tag{35}$$

Comparing (29) and (35) it is clear that for $i=0$ the quantities solving these expressions are the same so the allocations are identical regardless of the bonds' liquidity properties. For $i>0$ the right-hand side of (29) is larger than (35) because the marginal value of money is higher with illiquid outside bonds. The reason is that now, if an agent brings in money and does not need it, either because they are a seller or a low ε buyer, he can effectively trade the money for an interest bearing asset that compensates them for bringing 'idle' money into market 1. This increases the demand for money in market 3 and thus the real value of money balances. As a result, there is higher expected consumption in market 2. Thus, illiquid bonds will improve the allocation relative to the liquid bonds case if $i>0$. However, $i>0$ requires $\gamma>1$.

Figure 1 illustrates the point made here.[13] It compares steady-state welfare when bonds are illiquid or liquid for different inflation rates. At $\gamma=1$ the welfare levels are the same. For $\gamma>1$, the allocation with illiquid bonds dominates the one where bonds are illiquid. Note that welfare is increasing for low values of γ when bonds are illiquid. In the range where welfare is increasing the short-selling constraint is binding. We next explore under which condition it is optimal to set $\gamma>1$.

[13] Figure 1 is drawn for the utility function $u\left(q\right)=\left(1-\alpha\right)^{-1}q^{1-\alpha}$ with $\alpha=0.5$. We also used a uniform distribution of preference shocks on $[0,2]$ and set the measure of buyers to $n=0.5$ and the discount factor to $\beta=0.95$. We will use the same specification in all figures that follow. Finally, for Figure 1 we have assumed that the money/bonds ratio is $M_0/B_0=0.5$.

Figure 1: Illiquid vs Liquid Bonds



Is inflation welfare improving? Since illiquid outside bonds do not improve the allocation at $\gamma = 1$, is it optimal for the monetary authority to create inflation by setting $\gamma > 1$? Doing so has two effects. Raising γ above 1 makes illiquid bonds essential and allows those with low liquidity needs to be compensated for holding idle balances. This improves welfare. However, the higher inflation reduces real money balances, expected consumption and welfare. We would like to know under what conditions inflation is welfare improving.

Proposition 2 *Given M_0/B_0, there exists a critical value $\breve{\varepsilon} \in (0, \varepsilon_H)$, which is decreasing in M_0/B_0, such that positive steady state inflation is welfare improving if $\varepsilon_H > \tilde{\varepsilon} > \breve{\varepsilon}$.*

The intuition behind this result is the following. At $\gamma = 1$, $i = 0$ buyers with $\varepsilon \leq \tilde{\varepsilon}$ consume their first-best quantities while others do not. Consequently, if there is a sufficient measure of these agents ($\tilde{\varepsilon} > \breve{\varepsilon}$), then there is a first-order welfare gain from moving some consumption from those buyers at their first-best quantities to those who are not. However, buyers giving up consumption on the margin have to be compensated, which can be achieved by giving them interest bearing bonds. This requires raising γ marginally above one so that $i > 0$. Thus, for sufficiently large values of $\tilde{\varepsilon}$, the gain from this redistribution of consumption dominates the reduction in average consumption that occurs from the erosion of real money balances.[14] However, at a sufficiently high

[14] Note that a necessary condition for the welfare improving role of illiquid bonds is that there is more than one buyer type. With only one buyer type, in equilibrium, all buyers consume less than their first-best quantities for

inflation rate, $\gamma > \gamma_H$, then $\tilde{\varepsilon} = \varepsilon_H$ and no buyers are constrained on the margin since $\varepsilon u'(q) = 1+i$ for all ε. Thus, increasing inflation raises i lowering q for all buyers, which reduces welfare.

Figure 2: Inflation vs M/B

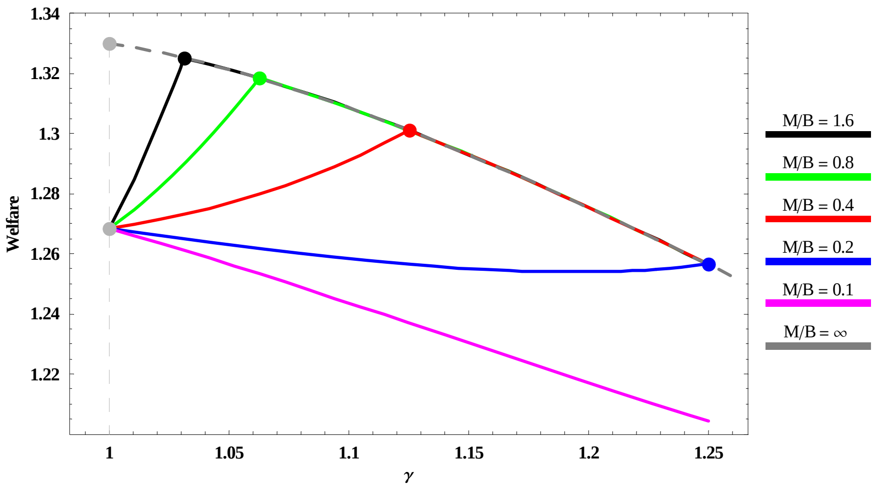


Figure 2 compares steady-state welfare in the economy with illiquid bonds for several values of the money/bonds ratio as a function of inflation γ. At $\gamma = 1$ all welfare levels are equal. The dots correspond to the inflation rates at which the economy switches from being constrained to unconstrained. One can see that the cutoff value $\breve{\varepsilon}$ is a function of the ratio M_0/B_0. For small values of M_0/B_0 (e.g. $M_0/B_0 = 0.1$ in Figure 2) we have $\breve{\varepsilon} > \tilde{\varepsilon}$ and so inflation is not welfare improving. For higher values (e.g. $M_0/B_0 = 0.4$) $\breve{\varepsilon}$ falls and the inequality reverses. Consequently, some inflation is beneficial. In this example, the dots for $M_0/B_0 = 0.4, 0.8, 1.6$ also corresponds to the optimal inflation rate for each economy. It is evident that increasing the ratio M_0/B_0 is never welfare decreasing. We discuss the optimal choice of M_0/B_0 below.

Optimal choice of M_0/B_0 From (31) there are two factors that affect i: the inflation rate and the relative supply of money to nominal bonds. Any optimal allocation requires that the marginal utility of consumption is equalized across all buyers. This requires that the left-hand side of (33) equals zero implying

$$\gamma = 1 + \frac{B_0}{M_0}(1-\beta) \tag{36}$$

$\gamma = 1$. Consequently, no beneficial redistribution of consumption can occur and inflation merely lowers the quantities consumed for all buyers, thus lowering welfare.

Using this expression and (32) we get

$$\varepsilon u'\left(q_\varepsilon\right) = \gamma/\beta \quad \forall\varepsilon.$$

It is easy to see that the optimal policy is to choose γ to be as small as possible yet keep it above one. Thus, from (36) this can be achieved by letting $B_0 \to 0$ implying γ is arbitrarily close to one. The reason for having B_0 being very small is similar to the arguments given by Kocherlakota (2003). The intuition is that if the stock of outside bonds is low, the government needs small amounts of money to redeem them. This means that it can keep the inflation rate low while still keeping the interest rate positive. To summarize, under the optimal policy the limiting allocation satisfies

$$\varepsilon u'\left(q_\varepsilon\right) = 1/\beta \quad \forall\varepsilon. \tag{37}$$

Thus, when the money-bond ratio is also a policy instrument, the optimal policy makes the nominal interest rate equals the real interest rate.

The optimal policy is depicted in Figure 2. From (36), for any ratio M_0/B_0 there is an optimal inflation rate γ. If we move in tandem the ratio M_0/B_0 and the inflation rate γ, increasing the former and decreasing the latter optimally, we trace up the dashed grey curve in Figure 2. In the limit we obtain the welfare level characterized by the grey dot in Figure 2. This welfare level is strictly higher than the welfare level obtained when bonds are liquid or if there are no bonds at all.

Why does an increase in M_0/B_0 improve welfare? For a given value of γ, an increase in M_0/B_0 means that for the same amount of money creation, more money must be paid out to a smaller stock of maturing nominal bonds. This means the nominal interest rate increases (see (31) to verify this). The increase in the nominal interest rate has two positive welfare effects and a negative welfare effect. First, sellers and low ε buyers receive a higher compensation for carrying idle balances. This raises the demand for money and thus its real value. Second, as i increases, fewer buyers are constrained, thereby reducing the inefficiency associated with $\varepsilon u'(\tilde{q}) > 1 + i$. On the downside, an increase in i reduces consumption of low ε buyers. However, because these buyers are buying small quantities of goods, the welfare loss is small. So, on net, the first two effects dominate and welfare increases in M_0/B_0.

Finally, one might ask: how can illiquid bonds rise the level of welfare even though the quantity in the limiting case approaches zero? In particular, how can they raise the purchasing power of the agents to such an extend? The trick to understand this is to see that the presence of illiquid bonds raises the demand for money and hence it's value even if the quantity is negligible small. This allows buyers to consume more and thereby improving the allocation in a non-negligible way.

4 Inside Bonds

In this section we analyze the model where there are no outside bonds but inside bonds can be traded in market 1. In market 1, sellers and low ε buyers can use their idle cash balances to acquire nominal bonds from the financial intermediary, which are redeemed in market 3. High ε buyers can issue nominal bonds in market 1 to the financial intermediary and redeem them in market 3. Inside bonds are perfectly divisible and one inside bond pays off $1+i$ units of fiat currency in market 3. Again, we focus on symmetric and stationary equilibria where all agents follow identical strategies and where real allocations are constant over time. In a stationary equilibrium end-of-period real money balances are time-invariant.

Since inside bonds are not held across periods, let $V_1(m)$ denote the expected value from entering market 1 with m units of money, $V_{2j}(m,b)$, $j=\varepsilon,s$, the expected value from entering market 2 with m units of money and b inside bonds, and $V_3(m,b)$ the expected value from entering market 3 with m and b.

In the third market, the problem of a representative agent is:

$$V_3\left(m,b\right) = \max_{x,h,m_{+1}} U\left(x\right) - h + \beta V_1\left(m_{+1}\right)$$
$$s.t. \quad x + \phi m_{+1} = h + \phi m + \phi b\left(1+i\right).$$

again yielding the first-order conditions $U'\left(x\right)=1$, $\phi = \beta V_1^m\left(m_{+1}\right)$ and the envelope conditions $V_3^m = \phi$ and $V_3^b = \phi\left(1+i\right)$.

In the second market, the agents' problems are unaffected by the types of bonds they hold so (12)-(13) are unaffected. Using the envelope theorem, the marginal values of bonds and the marginal values of money for buyers and sellers at the beginning of the second market are

$$V_{2\varepsilon}^{b} = V_{2s}^{b} = \phi(1+i) \tag{38}$$

$$V_{2\varepsilon}^{m} = \phi\varepsilon u'(q_{\varepsilon}) \text{ and } V_{2s}^{m} = \phi. \tag{39}$$

The first market The first market differs somewhat from the case with outside bonds so we discuss it in more detail. Let b_j denote the quantity of inside bonds purchased by an agent of type $j = \varepsilon, s$ in market 2. If $b_j < 0$, the agent is issuing inside bonds, i.e., he is borrowing. Note that there is only one short-selling constraint: agents cannot sell more money than they hold. However, because of the possibility of default agents cannot borrow more than $\bar{b}$, which implies the constraint $b \geq -\bar{b}$. Agents take this constraint as given. However, in equilibrium the upper bound $\bar{b}$ is determined endogenously.

An agent who holds m units of money at the opening of the first market has expected lifetime utility

$$\begin{aligned} V_1(m) &= (1-n)\int_0^{\varepsilon_H} V_{2\varepsilon}\left[m - b_\varepsilon, b_\varepsilon\right] dF(\varepsilon) \\ &\quad + nV_{2s}\left[m - b_s, b_s\right] \end{aligned} \tag{40}$$

where for $j = \varepsilon, s$ $\max_{b_j} V_{2j}\left[m - b_j, b_j\right]$ s.t. $m - b_j \geq 0$ and $b_j \geq -\bar{b}$. The first-order condition is

$$-V_{2j}^{m} + V_{2j}^{b} - \phi\mu_j + \phi\theta_j = 0 \tag{41}$$

where $\phi\mu_j$ is the Lagrange multiplier on the short-selling constraint $m - b_j \geq 0$ and $\phi\theta_j$ is the Lagrange multiplier on $b \geq -\bar{b}$. Obviously, both can not bind at the same time.

Consider first an agent who will be a producer in market 2. If $i < 0$, then $b_s = -\bar{b}$ and goods producers issue bonds to acquire money in market 1. This obviously cannot be an equilibrium and is ignored for the remainder of the paper which means that $\theta_s = 0$. If $i > 0$, it is optimal to spend your entire money holdings to acquire inside bonds. We can then use (14) and (15) to substitute V_{2j}^{m} and V_{2j}^{b} in (41) to get

$$\mu_s = i \tag{42}$$

If $i = 0$, a producer is indifferent on how much money to supply. Thus, a producer's bond demand is

$$\begin{aligned} &b_s \in \left[-\bar{b}, m\right] && \text{if} \quad i = 0 \\ &b_s = m && \text{if} \quad i > 0 \end{aligned} \tag{43}$$

Consider next an agent who will be a buyer in market 2. Since Inada conditions are assumed on $u(q)$ a buyer will always carry some money from market 1 to market 2. Thus, $\mu_\varepsilon = 0$. Accordingly, we can use (14) and (15) to write (41) as follows

$$\varepsilon u'(q_\varepsilon) = 1 + i + \theta_\varepsilon. \tag{44}$$

If $\varepsilon u'(q_\varepsilon) = 1 + i$, then $\theta_\varepsilon = 0$ and $b_\varepsilon > -\bar{b}$. Finally, if $\varepsilon u'(q_\varepsilon) > 1 + i$, then $\theta_\varepsilon > 0$ and $b_\varepsilon = -\bar{b}$. Thus, a buyer's bond demand is

$$\begin{aligned} &b_\varepsilon \in \left[-\bar{b}, m\right] && \text{if} \quad \varepsilon u'(q_\varepsilon) = 1 + i \\ &b_\varepsilon = -\bar{b} && \text{if} \quad \varepsilon u'(q_\varepsilon) > 1 + i. \end{aligned} \tag{45}$$

As was the case in the outside bonds economy, because a buyer's desired consumption is increasing in ε, there is a critical value for the taste index $\tilde{\varepsilon}$ such that

$$\tilde{\varepsilon} u'(\tilde{q}) = 1 + i \tag{46}$$

If $\varepsilon \leq \tilde{\varepsilon}$, he does not issue the maximal amount of inside bonds while if $\varepsilon \geq \tilde{\varepsilon}$ he issues the maximal amount and consumes

$$\tilde{q} = \phi m + \phi \bar{b}. \tag{47}$$

Accordingly, a buyer's consumption satisfies

$$q_\varepsilon = \begin{cases} u'^{-1}\left[(1+i)/\varepsilon\right] & \text{if} \quad \varepsilon \leq \tilde{\varepsilon} \\ u'^{-1}\left[(1+i)/\tilde{\varepsilon}\right] & \text{if} \quad \varepsilon \geq \tilde{\varepsilon} \end{cases} \tag{48}$$

Finally, apply the envelope theorem to equation (40) and using (15) the marginal value of money satisfies

$$\frac{\partial V_1(m,b)}{\partial m} = (1-n)\int_0^{\varepsilon_H} \phi\left[\varepsilon u'(q_\varepsilon)+\mu_\varepsilon\right]dF(\varepsilon) + n\phi(1+\mu_s) \tag{49}$$

4.1 Stationary equilibria

To derive the symmetric stationary equilibrium we have to solve for the equilibrium quantities and prices. In any symmetric equilibrium $m = M_{-1}$. Then use (44), (48) and (49), to write $\phi = \beta V_1^m(m_{+1})$ as follows

$$\frac{\gamma-\beta(1+i)}{\beta(1+i)} = (1-n)\int_{\tilde{\varepsilon}}^{\varepsilon_H}\left(\frac{\varepsilon}{\tilde{\varepsilon}}-1\right)dF(\varepsilon). \tag{50}$$

Comparing (50) to (30) we see that the cutoff values and thus the allocations for the outside bond and inside bond economies will be the same if the nominal interest rate in each economy is the same. Hence, what needs to be determined is whether or not the nominal interest rates will differ across the two economies.

We now derive the value of $\bar{b}$. Since $\bar{b}$ is a nominal variable what we really want is the real value $\bar{\ell} \equiv \phi\bar{b}$. This quantity is the maximal real amount that an agent is willing to repay in the last market. For buyers entering the last market with no money, who redeem their bonds, the expected discounted utility in a steady state is

$$V_3(m,b) = \max_{x,h,m_{+1}} U(x) - h_\varepsilon + \beta V_1(m_{+1})$$

where h_ε is a buyer's production in market 3 if he redeems his bonds. A defaulting buyer's expected discounted utility is

$$\widehat{V}_3(m,b) = U(\widehat{x}) - \widehat{h}_\varepsilon + \beta\widehat{V}_1(\widehat{m}_{1,+1})$$

where the hat indicates the optimal choice by a defaulter. The real borrowing constraint makes the agent indifferent between redeeming his bonds or defaulting so that $V_3(m,b) = \widehat{V}_3(m,b)$.

Defaulter When no enforcement exits, agents must voluntarily redeem their bonds. The only punishment for default is permanent exclusion from the inside bond market. Let $\widehat{q}_\varepsilon$ denote the quantity purchased by an agent with preference shock ε who is excluded from the inside bond market. It is straightforward to show that the marginal value of money for a defaulter satisfies

$$\frac{\gamma-\beta}{\beta}=(1-n)\int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right]dF\left(\varepsilon\right) \tag{51}$$

while (50) continues to determine the value of money for a non-defaulter. Since an agent who defaults can only use the money balances he brings into the period to buy goods, then there is a critical value $\hat{\varepsilon}$ such that

$$\hat{\varepsilon}u'\left(\widehat{q}\right)=1 \tag{52}$$

His consumption is

$$\widehat{q}_\varepsilon=\begin{cases} q_\varepsilon^* & \text{if } \ \varepsilon\leq\hat{\varepsilon} \\ u'^{-1}\left(1/\hat{\varepsilon}\right) & \text{if } \ \varepsilon\geq\hat{\varepsilon}\end{cases} \tag{53}$$

which means that he consumes the first-best quantity q_ε^* for $\varepsilon\leq\hat{\varepsilon}$ and the same quantity $u'^{-1}\left(1/\hat{\varepsilon}\right)$ for all $\varepsilon\geq\hat{\varepsilon}$.

Real borrowing constraint Given a borrowing constraint there are two possibilities: 1) the borrowing constraint is nonbinding for all agents or 2) it binds for some agents. The following Lemma is used for the remainder of this section.

Lemma 1 *The real borrowing constraint is*

$$\bar{\ell}=\frac{\beta}{(1+i)\,(1-\beta)}\left[(1-n)\,\Psi\left(q_\varepsilon,\widehat{q}_\varepsilon\right)+\left(\frac{\gamma-\beta}{\beta}\right)\left(\hat{q}-Q\right)\right] \tag{54}$$

where

$$\Psi\left(q_\varepsilon,\hat{q}_\varepsilon\right)=\int_0^{\varepsilon_H}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right]dF\left(\varepsilon\right)-\int_0^{\varepsilon_H}\left[\varepsilon u\left(\hat{q}_\varepsilon\right)-\hat{q}_\varepsilon\right]dF\left(\varepsilon\right).$$

We can now define a monetary equilibrium with inside bonds.

Definition 2 *A monetary equilibrium with unconstrained borrowing is a set* $\left\{q_\varepsilon,\widehat{q}_\varepsilon,\bar{\ell},i,\hat{\varepsilon},\tilde{\varepsilon}\right\}$ *sat-*

isfying $\tilde{\varepsilon} \geq \varepsilon_H$, (48), (53), (54) and

$$\begin{aligned} 0 < \ell_H &< \bar{\ell} && (55) \\ \frac{\gamma - \beta}{\beta} &= i && (56) \\ \frac{\gamma - \beta}{\beta} &= (1-n) \int_{\hat{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\hat{\varepsilon}} - 1 \right) dF(\varepsilon) && (57) \end{aligned}$$

Equation (56) is obtained by using (46) in (50) while (57) results from substituting (52) in (51).

Definition 3 *A monetary equilibrium with constrained borrowing is a set $\left\{ q_\varepsilon, \widehat{q}_\varepsilon, \bar{\ell}, i, \hat{\varepsilon}, \tilde{\varepsilon} \right\}$ satisfying (48), (53), (54) and*

$$\begin{aligned} \bar{\ell} &= u'^{-1}\left[(1+i) / \tilde{\varepsilon} \right] - Q && (58) \\ \frac{\gamma - \beta(1+i)}{\beta(1+i)} &= (1-n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) && (59) \\ \frac{\gamma - \beta}{\beta} &= (1-n) \int_{\hat{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\hat{\varepsilon}} - 1 \right) dF(\varepsilon) && (60) \end{aligned}$$

Equation (58) comes from a credit constrained borrower's cash constraint in market 2 while (59) is derived using (48) for $\varepsilon \leq \tilde{\varepsilon}$ in (50) and $u'(q_\varepsilon) = u'(\tilde{q})$ for $\varepsilon > \tilde{\varepsilon}$. Note that from (59) and (60) since the right-hand sides are decreasing functions of ε that $\tilde{\varepsilon} > \hat{\varepsilon}$ if $i > 0$.

Proposition 3 *For a value $\bar{\beta}$ sufficiently close to one, if $\beta \in \left[\bar{\beta}, 1\right)$ then there is an $\hat{\imath} > 0$ such that:*

(i) If $i \geq \hat{\imath}$, then a unique monetary equilibrium with unconstrained borrowing exists.

(ii) If $0 < i < \hat{\imath}$, then a monetary equilibrium with constrained borrowing may exist.

(iii) If $i = 0$ a unique monetary equilibrium without borrowing exists.

Since $i = 0$ at $\gamma = 1$ inside bonds are not traded and the allocation is the same as the illiquid outside bonds allocation at $i = 0$ at $\gamma = 1$.

4.2 Is inflation welfare improving?

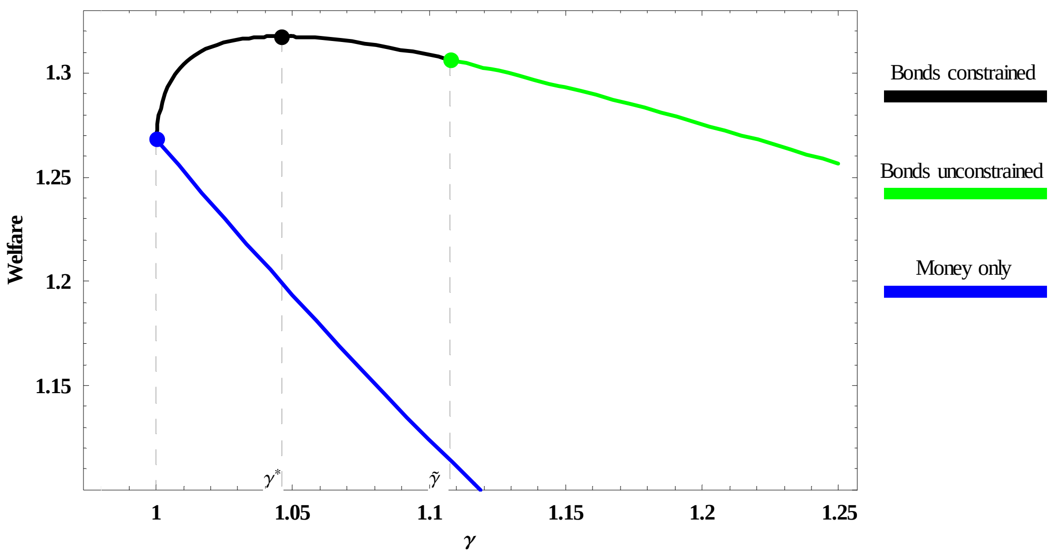
In an unconstrained borrowing equilibrium, it is straightforward to show that inflation is always welfare reducing since it reduces the real value of money balances and consumption for all agents.

However, in a constrained borrowing equilibrium, it may be optimal for the monetary authority to set $\gamma > 1$. As was the case with outside bonds, at $\gamma = 1$ and $i = 0$ some buyers consume their first-best quantities while others do not. Consequently, there is a first-order welfare gain from moving some consumption from those buyers at their first-best quantities to those who are not. In addition to this welfare gain, there is another positive welfare effect from raising γ above 1 – it increases the cost of being excluded from the banking system. This relaxes the borrowing constraint and creates a first-order welfare gain. However, the higher inflation reduces real money balances and expected consumption, which lowers welfare.[15] We can thus state the following

Proposition 4 *In a constrained borrowing equilibrium, if* $\beta > \left[1 + n + (1-n)\int_0^{\tilde{\varepsilon}} dF(\tilde{\varepsilon})\right]^{-1}$*, then a positive steady state inflation rate maximizes welfare* $\forall \tilde{\varepsilon}$.

Propositions 3 and 4 are illustrated in Figure 3. It displays welfare as function of γ for the economy with inside bonds and an economy without bonds. One can see that the welfare levels are equal at $\gamma = 0$. Then for $\gamma \in (0, \gamma^*)$ welfare is increasing in γ, and for $\gamma > \gamma^*$ it is decreasing. The grey dot represents the inflation rate $\tilde{\gamma}$ at which the economy switches from being constrained to unconstrained.

Figure 3: Inside Bonds



[15] BCW show that for a *degenerate* distribution for ε, inflation is always welfare increasing for sufficiently high values of β. In this section we extend those results to the case of a non-degenerate distribution of ε.

5 Inside vs outside bonds

In what follows we compare the economy with inside bonds with the one with outside bonds. For this comparison two facts are key. First, at $\gamma = 1$ the allocations in both economies are the same. Second, for sufficiently high inflation rates (34) and (56) are the same so the allocation in an unconstrained borrowing equilibrium is the same as the illiquid outside bond allocation when γ is sufficiently high. Thus, differences arise for low inflation rates only.

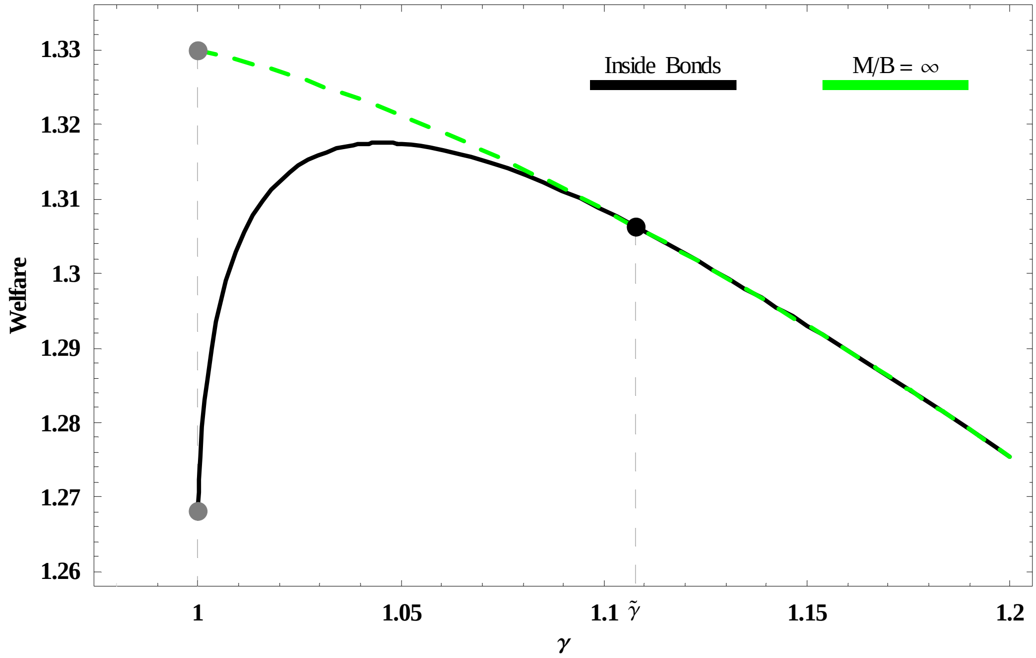
Proposition 5 *Any allocation attained in the economy with inside bonds can be replicated in the economy with outside bonds. The converse is not true.*

The proof of Proposition 5 is as follows. Consider any allocation in the economy with inside bonds for some γ^I. This allocation is characterized by some interest rate, say i^I. Choose the same inflation $\gamma = \gamma^I$ for the economy with outside bonds and some ratio M/B. From (31), this yields some interest rate i. Then, by changing M/B, one can attain the same interest rate $i = i^I$ as in the inside bonds economy. Since γ and i are the same in both economies, from (30) and (59), the cutoff values $\tilde{\varepsilon}$ are also the same. It then follows, from (24) and (48), that all quantities q_ε are the same. The converse is not true, since there are allocations in the outside bonds that cannot be attained in the inside bonds economy. This is illustrated in Figure 4.

Corollary 1 *The best allocation with outside bonds dominates the best allocation with inside bonds.*

Corollary 1 is illustrated in Figure 4. The dashed line plots steady-state welfare in the outside bonds economy for different values of γ when $M_0/B_0 \to \infty$. It is clear that welfare is strictly decreasing in γ and the optimal value of γ is positive but infinitesimally close to 1. The solid line plots welfare in the inside bonds economy without enforcement. It is clear that when borrowing constraints bind, welfare is strictly lower than in the outside bond economy.

Figure 4: Outside bonds dominate inside bonds



What is the intuition for this result? In the inside bond economy, there is only one instrument, γ, to solve two inefficiencies – the intensive margin of consumption and the distribution of consumption across agents. Furthermore, γ affects the individual decisions to default. In the outside bond economy where M_0/B_0 is endogenously chosen means the government now has two instruments to deal with two inefficiencies and individual default is not an issue. Thus, it is not surprising that the allocation with outside bonds dominates inside bonds in this case.

5.1 Enforcement of inside bonds

We now depart from one of our main assumptions by assuming that in the inside bonds economy redemption of inside bonds can be forced on agents. This allows us to state an equivalence result. When redemption can be forced, default is not feasible and agents have no borrowing constraints. It is straightforward to show that in a stationary equilibrium, the interest rate and all of the quantities

q_ε are determined by[16]

$$\begin{aligned} \frac{\gamma - \beta}{\beta} &= i & (61) \\ \varepsilon u'(q_\varepsilon) &= 1 + i & (62) \end{aligned}$$

The first equation comes from the agent's decision of how much money to bring into the period. In equilibrium they are indifferent between acquiring an additional unit of money in the centralized market or borrowing it in the inside bond market. The second equation comes from the buyers' decisions of how much to borrow. Since q_ε is unambiguously decreasing in i, it follows that $\gamma = 1$ generates the best allocation. This immediately implies that the best allocation with enforcement satisfies

$$\varepsilon u'(q_\varepsilon) = 1/\beta \quad \forall \varepsilon. \tag{63}$$

This immediately implies the following:

Proposition 6 *Assume repayment of loans can be enforced in the inside bonds economy. Then, the best allocation that can be attained in the economy with outside bonds can be replicated in the economy with inside bonds.*

The proof follows from comparing (**??**) with (63).

6 Conclusion

When agents are liquidity constrained, two options exist to relax this constraint: sell assets or issue debt. We have analyzed and compared the welfare properties of these two options in a model where agents can either issue nominal inside bonds or sell nominal outside bonds. The key assumption of our analysis is the absence of collection powers by private agents and the government. The following results emerged from our analysis. First, for any positive inflation rate, bonds are essential in both economies, and thus generate societal benefits. Second, we showed that for sufficiently high inflation rates, the allocations in the two economies are the same but they differ for low inflation rates. Third, any allocation attained in the economy with inside bonds can be replicated in the economy with

[16] See BCW (2006) for the derivation.

outside bonds. The converse is not true. Finally, under the optimal policies, the optimal allocation with illiquid outside bonds dominates the allocation with inside bonds.

7 Appendix

Proof of Proposition 1. Consider $\gamma < \bar{\gamma}$. From (33) define

$$f(\tilde{\varepsilon}) \equiv (1-n)\frac{1}{\tilde{\varepsilon}}\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) - (1-n)\int_{\tilde{\varepsilon}}^{\varepsilon_H} dF(\varepsilon).$$

We have $f'(\tilde{\varepsilon}) < 0$ with $\lim_{\tilde{\varepsilon}\to 0} f(\tilde{\varepsilon}) = +\infty$ and $f(\varepsilon_H) = 0$ so if $1 + (1-\gamma) M_0/B_0 \geq \beta$ or $\gamma_H = 1 + (1-\beta) B_0/M_0 \geq \gamma$ then a unique $0 < \tilde{\varepsilon} \leq \varepsilon_H$ solves (33). Otherwise, for $\bar{\gamma} > \gamma > \gamma_H$ we have $\tilde{\varepsilon} > \varepsilon_H$ and

$$q_\varepsilon = u'^{-1}\left(\frac{\gamma}{\varepsilon\left[1 + (1-\gamma)\frac{M_0}{B_0}\right]}\right)$$

for all ε. ■

Proof of Proposition 2. Substitute (2) into (1) and differentiate $(1-\beta)\mathcal{W}$ with respect to γ to get

$$(1-\beta)\left.\frac{d\mathcal{W}}{d\gamma}\right|_{\gamma=1} = (1-n)\int_0^{\varepsilon_H}\left[\varepsilon u'(q_\varepsilon) - 1\right]\left.\frac{dq_\varepsilon}{d\gamma}\right|_{\gamma=1} dF(\varepsilon) > 0.$$

Note first that for all $\varepsilon \leq \tilde{\varepsilon}$, $\varepsilon u'(q_\varepsilon) - 1 = 0$ at $\gamma = 1$ since $q_\varepsilon = q_\varepsilon^*$. It remains to show that $\left.\frac{dq_\varepsilon}{d\gamma}\right|_{\gamma=1} > 0$ for all $\varepsilon > \tilde{\varepsilon}$ since $\varepsilon u'(q_\varepsilon) - 1 > 0$ for all $\varepsilon > \tilde{\varepsilon}$. Since $q_\varepsilon = q_{\tilde{\varepsilon}}$ for all $\varepsilon \geq \tilde{\varepsilon}$ it is sufficient to show that $\left.\frac{dq_{\tilde{\varepsilon}}}{d\gamma}\right|_{\gamma=1} > 0$.

Totally differentiate (32)

$$u'(\tilde{q})d\tilde{\varepsilon} + \tilde{\varepsilon}u''(\tilde{q})d\tilde{q} = \frac{1 + (1-\gamma)\frac{M_0}{B_0} + \gamma\frac{M_0}{B_0}}{\left[1 + (1-\gamma)\frac{M_0}{B_0}\right]^2} d\gamma$$

so

$$\frac{d\tilde{q}}{d\gamma} = \frac{-1}{\tilde{\varepsilon}u''(\tilde{q})}\left\{u'(\tilde{q})\frac{d\tilde{\varepsilon}}{d\gamma} - \frac{1}{\left[1 + (1-\gamma)\frac{M_0}{B_0}\right]^2}\left(1 + \frac{M_0}{B_0}\right)\right\} \tag{64}$$

Totally differentiate (33)

$$\begin{aligned} -\frac{M_0}{\beta B_0}d\gamma &= -(1-n)\frac{1}{\tilde{\varepsilon}^2}\int_{\tilde{\varepsilon}}^{\varepsilon_H}\varepsilon dF(\varepsilon)\, d\tilde{\varepsilon} \\ \frac{d\tilde{\varepsilon}}{d\gamma} &= \frac{M_0}{\beta B_0(1-n)}\frac{\tilde{\varepsilon}^2}{\int_{\tilde{\varepsilon}}^{\varepsilon_H}\varepsilon dF(\varepsilon)} > 0 \end{aligned}$$

Substitute into (64) to get

$$\frac{d\tilde{q}}{d\gamma} = \frac{-1}{\tilde{\varepsilon}u''(\tilde{q})}\left\{u'(\tilde{q})\frac{M_0}{\beta B_0 (1-n)}\frac{\tilde{\varepsilon}^2}{\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)} - \frac{1}{\left[1+(1-\gamma)\frac{M_0}{B_0}\right]^2}\left(1+\frac{M_0}{B_0}\right)\right\}$$

Evaluate at $\gamma = 1$ and $\tilde{\varepsilon}u'(\tilde{q}) = 1$

$$\left.\frac{d\tilde{q}}{d\gamma}\right|_{\gamma=1} = \frac{-1}{\tilde{\varepsilon}u''(\tilde{q})}\frac{M_0}{B_0}\left[\frac{\tilde{\varepsilon}}{\beta(1-n)\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)} - 1 - \frac{B_0}{M_0}\right]$$

which is positive if

$$\frac{\tilde{\varepsilon}}{\beta(1-n)\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)} > 1 + \frac{B_0}{M_0} \equiv \bar{\gamma}$$

Define

$$z(\tilde{\varepsilon}) \equiv \frac{\tilde{\varepsilon}}{\beta(1-n)\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)}$$

with

$$z'(\tilde{\varepsilon}) \equiv \frac{\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) + \tilde{\varepsilon}^2 dF(\tilde{\varepsilon})}{\beta(1-n)\left[\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)\right]^2} > 0$$

Note that $z(\tilde{\varepsilon})$ is continuous with $z(0) = 0$ and $z(\varepsilon_H) \to +\infty$. Thus, a unique value $\breve{\varepsilon}$ solves $z(\breve{\varepsilon}) = \bar{\gamma}$ so if $\tilde{\varepsilon} > \breve{\varepsilon}$ then inflation is welfare improving. ■

Proof of Lemma 1. Consider a borrower who borrowed $-\bar{b}$ in market 1 and is considering defaulting on his issued bonds in market 3.

If he redeems his bonds, he gets the equilibrium expected discounted utility in a steady state is

$$V_3(m,b) = U(x^*) - h_\varepsilon + \beta V_1(m_{1,+1})$$

where h_ε is his production in the market 3 if he repays his loan. A defaulter's expected discounted utility is

$$\widehat{V}_3(m,b) = U(\widehat{x}) - \widehat{h}_\varepsilon + \beta \widehat{V}_1(\widehat{m}_{1,+1})$$

where the hat indicates the optimal choice by a defaulter. The real borrowing constraint $\bar{\ell} = \phi\bar{b}$

satisfies $V_3(m,b) = \widehat{V}_3(m,b)$ or

$$U(x^*) - h_\varepsilon + \beta V(m_{1,+1}) = U(\widehat{x}) - \widehat{h}_\varepsilon + \beta \widehat{V}(\widehat{m}_{1,+1})$$

Since $\widehat{x} = x^*$ we have $h_\varepsilon - \widehat{h}_\varepsilon = \beta\left[V(m_{1,+1}) - \widehat{V}(\widehat{m}_{1,+1})\right]$. The continuation payoffs are

$$\begin{aligned}
(1-\beta)\widehat{V}(\widehat{m}_{1,+1}) &= \int_0^{\varepsilon_H}\left[(1-n)\varepsilon u(\widehat{q}_\varepsilon) - n\widehat{q}_s\right]dF(\varepsilon) + U(x^*) - \widehat{h} \\
(1-\beta)V(m_{1,+1}) &= \int_0^{\varepsilon_H}\left[(1-n)\varepsilon u(q_\varepsilon) - nq_s\right]dF(\varepsilon) + U(x^*) - h.
\end{aligned}$$

where h is expected hours worked for a nondefaulter. Accordingly, we have

$$h_\varepsilon - \widehat{h}_\varepsilon = \frac{\beta}{1-\beta}\left[(1-n)\Psi(q_\varepsilon, \widehat{q}_\varepsilon) + \widehat{h} - h\right]. \tag{65}$$

where $\Psi(q_\varepsilon, \widehat{q}_\varepsilon) = \int_0^{\varepsilon_H}\left[\varepsilon u(q_\varepsilon) - q_\varepsilon\right]dF(\varepsilon) - \int_0^{\varepsilon_H}\left[\varepsilon u(\widehat{q}_\varepsilon) - \widehat{q}_\varepsilon\right]dF(\varepsilon) > 0$. We get this expression for $\Psi(q_\varepsilon, \widehat{q}_\varepsilon)$ because costs are linear sellers are indifferent as to how much they produce thus we just assume that the deviator produces in the decentralized market $\hat{q}_s = \left(\frac{1-n}{n}\right)\int_{\varepsilon_L}^{\varepsilon_H}\widehat{q}_\varepsilon dF(\varepsilon)$ in each period. Furthermore, from market clearing we have $nq_s = (1-n)\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$.

Deriving $h_\varepsilon - \widehat{h}_\varepsilon$: If the buyer redeems his bonds he works

$$\begin{aligned}
h_\varepsilon &= x^* + \phi m_{1,+1} - \phi\left[m_1 + \bar{b} - pq_\varepsilon\right] - \phi\tau M_{-1} + \phi\bar{b}(1+i) \\
&= x^* + \phi m_{1,+1} - \phi\left[m_1 - pq_\varepsilon\right] - \phi\tau M_{-1} + \phi\bar{b}i \\
&= x^* + q_\varepsilon + \bar{\ell}i
\end{aligned}$$

where we use the equilibrium condition $m_{1,+1} = m_1 + \tau M_{-1} = \gamma m_1$. If he defaults on his bonds, he works

$$\begin{aligned}
\widehat{h}_\varepsilon &= x^* + \phi\widehat{m}_{1,+1} - \phi\left[m_1 + \bar{b} - pq_\varepsilon\right] - \phi\tau M_{-1} \\
&= x^* + \phi(\widehat{m}_{1,+1} - m_{1,+1}) - \bar{\ell} + q_\varepsilon \\
&= x^* + \phi\gamma(\widehat{m}_1 - m_1) - \bar{\ell} + q_\varepsilon
\end{aligned}$$

where we use the equilibrium condition that a defaulter's money balances must grow at the rate γ

so $\widehat{m}_{1,+1} = \gamma \widehat{m}_1$. Thus

$$h_\varepsilon - \widehat{h}_\varepsilon = (1+i)\,\bar{\ell} - \phi\gamma\,(\widehat{m}_1 - m_1) \tag{66}$$

Deriving $\widehat{h} - h$: Once the agent defaults, as a buyer he spent $p\widehat{q}_\varepsilon$ units of money so his hours worked are

$$\begin{aligned}
\widehat{h}_\varepsilon &= x^* + \phi\widehat{m}_{1,+1} - \phi\,(\widehat{m}_1 - p\widehat{q}_\varepsilon) - \phi\tau M_{-1} \\
&= x^* + \phi\,(\widehat{m}_{1,+1} - \widehat{m}_1) + \phi p\widehat{q}_\varepsilon - \phi\,(m_{1,+1} - m_1) \\
&= x^* + (\gamma - 1)\,\phi\,(\widehat{m}_1 - m_1) + \phi p\widehat{q}_\varepsilon
\end{aligned}$$

For a seller we have

$$\begin{aligned}
\widehat{h}_s &= x^* + \phi\widehat{m}_{1,+1} - \phi\,(\widehat{m}_1 + p\widehat{q}_s) - \phi\tau M_{-1} \\
&= x^* + (\gamma - 1)\,\phi\,(\widehat{m}_1 - m_1) - \phi p\left(\frac{1-n}{n}\right)\int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF\,(\varepsilon)
\end{aligned}$$

So for a defaulter expected hours worked are

$$\widehat{h} = (1-n)\,\widehat{h}_\varepsilon + n\widehat{h}_s = x^* + (\gamma - 1)\,\phi\,(\widehat{m}_1 - m_1)$$

while if he does not deviate he works $h = x^*$ and so

$$\widehat{h} - h = (\gamma - 1)\,\phi\,(\widehat{m}_1 - m_1) \tag{67}$$

Solving for $\bar{\ell}$: Using (65)-(67) we get

$$\bar{\ell} = \frac{\beta}{(1-\beta)\,(1+i)}\left[(1-n)\,\Psi\,(q_\varepsilon, \widehat{q}_\varepsilon) + \left(\frac{\gamma-\beta}{\beta}\right)(\phi\widehat{m}_1 - \phi m_1)\right]$$

In equilibrium $\phi\widehat{m}_1 = \widehat{q}$, and using market clearing $\phi m_1 = \phi M_{-1} = Q \equiv (1-n)\int_0^{\varepsilon_H} q_\varepsilon dF\,(\varepsilon)$. So we have

$$\bar{\ell} = \frac{\beta}{(1-\beta)\,(1+i)}\left[(1-n)\,\Psi\,(q_\varepsilon, \widehat{q}_\varepsilon) + \left(\frac{\gamma-\beta}{\beta}\right)(\widehat{q} - Q)\right]$$

To know whether $\bar{\ell} > 0$ we need to determine the sign of right hand side. Substituting for $\Psi\,(q_\varepsilon, \widehat{q}_\varepsilon)$

and Q we need

$$(1-n)\int_0^{\varepsilon_H}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right]dF\left(\varepsilon\right)-(1-n)\int_0^{\varepsilon_H}\left[\varepsilon u\left(\widehat{q}_\varepsilon\right)-\widehat{q}_\varepsilon\right]dF\left(\varepsilon\right)+$$
$$\left(\frac{\gamma-\beta}{\beta}\right)\left[\widehat{q}-(1-n)\int_0^{\varepsilon_H}q_\varepsilon dF\left(\varepsilon\right)\right]>0$$

We have

$$\frac{\gamma-\beta}{\beta}=(1-n)\int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right]dF\left(\varepsilon\right) \tag{68}$$

Substitute in to get

$$\begin{aligned}(1-n)\int_0^{\varepsilon_H}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right]dF\left(\varepsilon\right)-(1-n)\int_0^{\varepsilon_H}\left[\varepsilon u\left(\widehat{q}_\varepsilon\right)-\widehat{q}_\varepsilon\right]dF\left(\varepsilon\right)&\\ +(1-n)\int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right]dF\left(\varepsilon\right)\left[\widehat{q}-(1-n)\int_0^{\varepsilon_H}q_\varepsilon dF\left(\varepsilon\right)\right]&>0\end{aligned}$$

$$\begin{aligned}\int_0^{\varepsilon_H}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right]dF\left(\varepsilon\right)-\int_0^{\varepsilon_H}\left[\varepsilon u\left(\widehat{q}_\varepsilon\right)-\widehat{q}_\varepsilon\right]dF\left(\varepsilon\right)&\\ +\int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right]dF\left(\varepsilon\right)\left[\widehat{q}-(1-n)\int_0^{\varepsilon_H}q_\varepsilon dF\left(\varepsilon\right)\right]&>0\end{aligned}$$

In an unconstrained borrowing equilibrium we have

$$\frac{\gamma-\beta}{\beta}=\int_0^{\varepsilon_H}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right]dF\left(\varepsilon\right) \tag{69}$$

So (68) and (69) yield

$$1-n=\frac{\int_0^{\varepsilon_H}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right]dF\left(\varepsilon\right)}{\int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right]dF\left(\varepsilon\right)}<1$$

Substitute this in

$$\begin{aligned}\int_0^{\varepsilon_H}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right]dF\left(\varepsilon\right)-\int_0^{\varepsilon_H}\left[\varepsilon u\left(\widehat{q}_\varepsilon\right)-\widehat{q}_\varepsilon\right]dF\left(\varepsilon\right)&\\ +\int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right]dF\left(\varepsilon\right)\left[\widehat{q}-\frac{\int_0^{\varepsilon_H}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right]dF\left(\varepsilon\right)}{\int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right]dF\left(\varepsilon\right)}\int_0^{\varepsilon_H}q_\varepsilon dF\left(\varepsilon\right)\right]&>0\end{aligned}$$

$$\begin{aligned} & \int_0^{\varepsilon_H} \left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right] dF(\varepsilon)-\int_0^{\varepsilon_H}\left[\varepsilon u\left(\widehat{q}_\varepsilon\right)-\widehat{q}_\varepsilon\right] dF(\varepsilon) \\ +\widehat{q} \int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right] dF(\varepsilon)-\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) \int_0^{\varepsilon_H}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right] dF(\varepsilon) & > 0 \end{aligned}$$

Rewrite as

$$\begin{aligned} & \int_0^{\varepsilon_H}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right] dF(\varepsilon)-\int_0^{\varepsilon_H}\left[\varepsilon u\left(\widehat{q}_\varepsilon\right)-\widehat{q}_\varepsilon\right] dF(\varepsilon) \\ > & \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) \int_0^{\varepsilon_H}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right] dF(\varepsilon)-\widehat{q} \int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right] dF(\varepsilon) \end{aligned}$$

Divide by sides by $\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)-\int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)$ to get

$$\begin{aligned} & \frac{\int_0^{\varepsilon_H}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right] dF(\varepsilon)-\int_0^{\varepsilon_H}\left[\varepsilon u\left(\widehat{q}_\varepsilon\right)-\widehat{q}_\varepsilon\right] dF(\varepsilon)}{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)-\int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)} \\ > & \int_0^{\varepsilon_H}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right] dF(\varepsilon) \frac{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)-\widehat{q} \frac{\int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right] dF(\varepsilon)}{\int_0^{\varepsilon_H}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right] dF(\varepsilon)}}{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)-\int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)} \end{aligned}$$

which always holds because the LHS is greater than $\int_0^{\varepsilon_H}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right] dF(\varepsilon)$ and

$$\frac{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)-\widehat{q} \frac{\int_0^{\varepsilon_H}\left[\varepsilon u'\left(\widehat{q}_\varepsilon\right)-1\right] dF(\varepsilon)}{\int_0^{\varepsilon_H}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right] dF(\varepsilon)}}{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)-\int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)}<1$$

So $\bar{\ell}>0$ in an unconstrained equilibrium. ■

Proof of Proposition 3. In an unconstrained equilibrium we have unique values for q_ε and $i=(\gamma-\beta)/\beta$. All that is left is to show is that $\ell_H \leq \bar{\ell}$ or

$$\ell_H \leq \frac{\beta}{(1-\beta)(1+i)}(1-n) \Psi\left(q_\varepsilon, \widehat{q}_\varepsilon\right)+\frac{\beta i(\widehat{q}-Q)}{(1-\beta)(1+i)}$$

Since all agents (including the one with ε_H) are unconstrained we have $\ell_H=q_{\varepsilon_H}-\phi M_{-1}$ and $\phi M_{-1}=Q$ so we have

$$(1-\beta)(1+i)\left(q_{\varepsilon_H}-Q\right) \leq \beta(1-n) \Psi\left(q_\varepsilon, \widehat{q}_\varepsilon\right)+i \beta(\widehat{q}-Q) \tag{70}$$

Define

$$\Delta\left(i,\beta\right) \equiv \left(1-\beta\right)\left(1+i\right)\left(q_{\varepsilon_H}-Q\right)-\beta\left(1-n\right)\Psi\left(q_\varepsilon,\widehat{q}_\varepsilon\right)-i\beta\left(\widehat{q}-Q\right)$$

So we need $\Delta\left(i,\beta\right)\leq 0$ in an unconstrained borrowing equilibrium. Note that $\Delta\left(0,\beta\right)=\left(1-\beta\right)\left(q^*_{\varepsilon_H}-Q^*\right)>0$ since $\Psi\left(q^*_\varepsilon,\widehat{q}^*_\varepsilon\right)=0$ at $i=0$. Thus, (70) is violated at the Friedman rule.

Consider solutions to $\Delta\left(i,\beta\right)=0$. Note that $\Delta(0,1)=0$ since $q_\varepsilon|_{(0,1)}=\widehat{q}_\varepsilon|_{(0,1)}=q^*_\varepsilon$ and $\Psi\left(q^*_\varepsilon,\widehat{q}^*_\varepsilon\right)=0$. Let $\Delta_i\left(i,\beta\right)\equiv\frac{\partial\Delta(i,\beta)}{\partial i}$. Then, we have

$$\begin{aligned}\Delta_i\left(i,\beta\right) &= \left(1-\beta\right)\left(q_{\varepsilon_H}-Q\right)+\left(1-\beta\right)\left(1+i\right)\left(\frac{\partial q_{\varepsilon_H}}{\partial i}-\frac{\partial Q}{\partial i}\right)\\ &\quad -\beta\left(1-n\right)\frac{\partial\Psi\left(q_\varepsilon,\widehat{q}_\varepsilon\right)}{\partial i}-\beta\left(\widehat{q}-Q\right)-\beta i\left(\frac{\partial\widehat{q}}{\partial i}-\frac{\partial Q}{\partial i}\right)\end{aligned}$$

Since the partial derivatives in this expression are all continuous $\Delta_i\left(i,\beta\right)$ is continuous and non-zero with

$$\Delta_i\left(0,1\right)=\left.-\beta\left(\widehat{q}-Q\right)\right|_{(0,1)}=-\left[q^*_{\varepsilon_H}-\left(1-n\right)\int_0^{\varepsilon_H}q^*_\varepsilon dF\left(\varepsilon\right)\right]<0.$$

since $\left.\frac{\partial\Psi(q_\varepsilon,\widehat{q}_\varepsilon)}{\partial i}\right|_{(0,1)}=0$ and $\widehat{q}|_{(0,1)}=q^*_{\varepsilon_H}$.

Let $\Delta_\beta\left(i,\beta\right)\equiv\frac{\partial\Delta(i,\beta)}{\partial\beta}$. Then, we have

$$\begin{aligned}\Delta_\beta\left(i,\beta\right) &= -\left(1+i\right)\left(q_{\varepsilon_H}-Q\right)+\left(1-\beta\right)\left(1+i\right)\left(\frac{\partial q_{\varepsilon_H}}{\partial\beta}-\frac{\partial Q}{\partial\beta}\right)\\ &\quad -\left(1-n\right)\Psi\left(q_\varepsilon,\widehat{q}_\varepsilon\right)-\beta\left(1-n\right)\frac{\partial\Psi\left(q_\varepsilon,\widehat{q}_\varepsilon\right)}{\partial\beta}-i\left(\widehat{q}-Q\right)-i\beta\left(\frac{\partial\widehat{q}}{\partial\beta}-\frac{\partial Q}{\partial\beta}\right)\end{aligned}$$

Therefore $\Delta_\beta\left(0,1\right)$ is continuous and

$$\Delta_\beta\left(0,1\right)=-\left[q^*_{\varepsilon_H}-\left(1-n\right)\int_0^{\varepsilon_H}q^*_\varepsilon dF\left(\varepsilon\right)\right]<0.$$

since $\left.\frac{\partial\Psi(q_\varepsilon,\widehat{q}_\varepsilon)}{\partial\beta}\right|_{(0,1)}=0$. By the implicit function theorem, it follows that, for β arbitrarily close to one, the expression $\Delta(i,\beta)=0$ defines i as an implicit function of β, i.e., $i=\hat{\imath}(\beta)$.

Furthermore, we have

$$\left.\frac{di}{d\beta}\right|_{(0,1)}=-\frac{\Delta_\beta\left(0,1\right)}{\Delta_i\left(0,1\right)}=-1,$$

so that as β falls i grows. It follows from the implicit function theorem that $\Delta(\hat{\imath},\beta)=0$ for a unique $\hat{\imath} > 0$ and β sufficiently close to one.

Establishing existence and uniqueness of the unconstrained credit equilibrium for $i > \hat{\imath}$. Above we established that $\Delta(0, \beta) > 0$ for all $0 < \beta < 1$. Fix β close to 1. We have established that $\Delta(\hat{\imath}, \beta) = 0$ for some $\hat{\imath} > 0$. By continuity, we have that if $i > \hat{\imath}$ then $\Delta(0, \beta) < 0$ and so an unconstrained equilibrium exists. For $0 \leq i < \hat{\imath}$, then $\Delta(i, \beta) \geq 0$ which violates (70). This establishes the first part of Proposition 3.

Consider $0 \leq i < \hat{\imath}$. In general we cannot prove existence or uniqueness. We now characterize the properties of (54) and (58)-(60). At $i = 0$, (59)-(60) imply $\tilde{\varepsilon} = \hat{\varepsilon}$ so $\tilde{q} = \widehat{q}$ and $\Psi(q_\varepsilon, \widehat{q}_\varepsilon) = 0$. Then from (54) and (58) we have $\gamma = 1$. This implies there is one and only one monetary policy consistent with a nominal interest rate of zero and also satisfies (58). Thus, a monetary equilibrium with credit does not exist at $\gamma = 1$.

Furthermore, we have

$$\left.\frac{di}{d\gamma}\right|_{\gamma=1} = \frac{1}{1-\beta} > 0 \tag{71}$$

To obtain this, use (58) to replace $\bar{\ell}$ in (54) and then totally differentiate the resulting expression:

$$(\tilde{q} - Q)(1+i) = \frac{\beta(1-n)}{1-\beta}\Psi(q_\varepsilon, \widehat{q}_\varepsilon) + \left(\frac{\gamma - \beta}{1-\beta}\right)(\widehat{q} - Q)$$

$$\begin{aligned}
&(1+i)(d\tilde{q} - dQ) + (\tilde{q} - Q)\,di \\
= \;& \frac{\beta(1-n)}{1-\beta}\left\{\left[\int_0^{\varepsilon_H} \varepsilon u'(q_\varepsilon)\,dq_\varepsilon - dq_\varepsilon\right] dF(\varepsilon) - \left[\int_0^{\varepsilon_H} \varepsilon u'(\hat{q}_\varepsilon)\,d\widehat{q}_\varepsilon - d\widehat{q}_\varepsilon\right] dF(\varepsilon)\right\} \\
&+\frac{\beta}{1-\beta}\left[\left(\frac{\gamma-\beta}{\beta}\right)(d\widehat{q} - dQ) + \frac{1}{\beta}(\widehat{q} - Q)\,d\gamma\right]
\end{aligned}$$

Evaluate at $i = 0$ and $\gamma = 1$ to get

$$\begin{aligned}
&(d\tilde{q} - dQ) + (\tilde{q} - Q)\,di \\
= \;& \frac{\beta(1-n)}{1-\beta}\left\{\left[\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon u'(q_\varepsilon) - 1\right] dF(\varepsilon)\,d\tilde{q} - \left[\int_{\hat{\varepsilon}}^{\varepsilon_H} \varepsilon u'(\hat{q}_\varepsilon) - 1\right] dF(\varepsilon)\,d\widehat{q}\right\} \\
&+\frac{\beta}{1-\beta}\left[\left(\frac{\gamma-\beta}{\beta}\right)(d\widehat{q} - dQ) + \frac{1}{\beta}(\widehat{q} - Q)\,d\gamma\right]
\end{aligned}$$

Note $\left[\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon u'(q_\varepsilon) - 1\right] dF(\varepsilon) = \left[\int_{\hat{\varepsilon}}^{\varepsilon_H} \varepsilon u'(\hat{q}_\varepsilon) - 1\right] dF(\varepsilon) = \frac{1-\beta}{\beta}$ and use $\tilde{q} = \widehat{q}$ and (59)-(60) to get

$$\begin{aligned} d\tilde{q} - dQ + (\widehat{q} - Q)\, di &= \frac{\beta}{1-\beta}\left(\frac{1-\beta}{\beta} d\tilde{q} - \frac{1-\beta}{\beta} d\widehat{q}\right) \\ &+ \frac{\beta}{1-\beta}\left[\left(\frac{1-\beta}{\beta}\right)(d\widehat{q} - dQ) + \frac{1}{\beta}(\widehat{q} - Q)\, d\gamma\right] \end{aligned}$$

This expression reduces to (71). ■

Proof of Proposition 4. In equilibrium, welfare is given by (1). Again, using (2) differentiate (1) with respect to γ to get

$$(1-\beta)\left.\frac{d\mathcal{W}}{d\gamma}\right|_{\gamma=1} = (1-n)\int_0^{\varepsilon_H} \left[\varepsilon u'(q_\varepsilon) - 1\right] \left.\frac{dq_\varepsilon}{d\gamma}\right|_{\gamma=1} dF(\varepsilon) > 0.$$

Since $\varepsilon u'(q_\varepsilon) - 1 = 0$ for all $\varepsilon \leq \tilde{\varepsilon}$ at $\gamma = 1$, and $q_\varepsilon = \tilde{q}$ for all $\varepsilon \geq \tilde{\varepsilon}$, welfare will be increasing in γ if $\left.\frac{d\tilde{q}}{d\gamma}\right|_{\gamma=1} > 0$.

Using $\varepsilon u'(q_\varepsilon) = 1 + i$ for all $\varepsilon \leq \tilde{\varepsilon}$ and $q_\varepsilon = \tilde{q}$ for all $\varepsilon \geq \tilde{\varepsilon}$ (50) can be written as

$$\frac{\gamma - \beta}{\beta} = (1-n)\, u'(\tilde{q}) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) + (1-n)(1+i)\int_0^{\tilde{\varepsilon}} dF(\varepsilon) - 1 + n + ni \tag{72}$$

Totally differentiate (72):

$$\begin{aligned} \frac{1}{\beta} d\gamma &= (1-n)\, u''(\tilde{q})\, d\tilde{q} \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) - (1-n)\left[\tilde{\varepsilon} u'(\tilde{q}) - (1+i)\right] dF(\tilde{\varepsilon})\, d\tilde{\varepsilon} \\ &+ \left[n + (1-n)\int_0^{\tilde{\varepsilon}} dF(\varepsilon)\right] di \end{aligned}$$

Using $\tilde{\varepsilon} u'(\tilde{q}) = 1 + i$ we have

$$\frac{1}{\beta} d\gamma = (1-n)\, u''(\tilde{q})\, d\tilde{q} \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) + \left[n + (1-n)\int_0^{\tilde{\varepsilon}} dF(\varepsilon)\right] di \tag{73}$$

Substituting (71) into (73) gives

$$\frac{1}{\beta} d\gamma = \left[(1-n)\, u''(\tilde{q}) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)\right] d\tilde{q} + \left[n + (1-n)\int_0^{\tilde{\varepsilon}} dF(\varepsilon)\right] \frac{1}{1-\beta} d\gamma$$

Thus

$$\left.\frac{d\tilde{q}}{d\gamma}\right|_{\gamma=1} = \frac{1-\beta-\beta\left[n+(1-n)\int_0^{\tilde{\varepsilon}} dF(\varepsilon)\right]}{\beta(1-\beta)(1-n)u''(\tilde{q})\int_{\tilde{\varepsilon}}^{\varepsilon_H}\varepsilon dF(\varepsilon)}$$

The denominator is negative. So $\left.\frac{d\tilde{q}}{d\gamma}\right|_{\gamma=1} > 0$ and $(1-\beta)\left.\frac{dW}{d\gamma}\right|_{\gamma=1} > 0$ if

$$\beta > \frac{1}{1+n+(1-n)\int_0^{\tilde{\varepsilon}} dF(\varepsilon)}.$$

■

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