A Model of Small Change Shortages

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Abstract

There are numerous historical episodes in which agents have complained about there being a shortage of small coins (low value media of exchange). In the paper, we build a pairwise matching model with two indivisible commodity monies. Because the monies are indivisible, the model takes seriously the idea that these monies are coins. These commodity monies have different intrinsic values, and agents may hold more than one unit of one or both. We find equilibria in which both monies are used for trades and in which the lower intrinsic-valued money is used to make change in some transactions.

Keywords: commodity money, random matching, Gresham's Law, optimal denominations
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1 Introduction

There are numerous historical episodes in which agents have complained about there being a shortage of small coins (low value media of exchange). For example, Spufford (1970) states that there was a " continual demand for small change, the lack of which was frequent topic of popular complaint" in the Burgundian Netherlands in the 15th century. According to Sargent and Velde (2002) in France "In 1337, a debasement of 33% was justified by 'the peoples's suffering and lack of coins,' and the same phrase was used on the occasion of another debasement in April 1340."(2002, p. 135)

Cipolla (1956) called these shortages "the big problem of small change." According to Sargent and Velde (2002), this problem was "the intermittent shortages of small denomination coins, persistent depreciations of small coins relative to large ones, and recurrent debasements of the small coins." (p. 5)

The lack of small change appears to have caused the following problem. In order to complete a purchase, a potential buyer would either have to give over more value than would have been the case if a smaller value coin would have been available or else forgo the transaction altogether. This is very well illustrated by a great quote taken from Ruding and Akerman (1840) by Sargent and Velde (2002, p. 134) about trading difficulties in England in the 1400s:

> Men travelling [sic] over countries, for part of their expenses of necessity must depart our sovereign lord's coin, that is to wit, a penny in two pieces, or else forego all the same penny, for the payment of a half penny; and also the poor common retailers of victuals, and of other needful things, for default of such coin of half pennies and farthings, oftentimes may not sell their said victuals and things, and many of our said sovereign lord's poor liege people, which would buy such victuals and other small things necessary, may not buy them, for default of half pennies and farthings not had on the part of the buyer nor on the part of the seller; which scarcity of half pennies and farthings, has fallen, and daily yet does, because that for their great weight, and the fineness of allay, they be daily tried and molten, and put into other use, unto the increase of winning of them that do so.

Sargent and Velde (2002) (hereafter SV) build a model that delivers small change shortages. The model has two goods and two monies, which are to be thought of as being a large coin, which they call a "dollar", and a small coin, which they call a "penny." One of the goods can be purchased with either of the two coins. The other good can only be purchased with pennies, however. In other words, in addition to the usual budget constraint, the model has a "penny-in-advance" (PIA) constraint.

A small change shortage is the case in which the PIA constraint binds and the Lagrangean multiplier on this constraint is positive. The bindingness of the PIA constraint gives pennies a additional implicit rate of return over dollars. As a consequence, dollars must appreciate in value relative to pennies in order for them to be held. This causes the price level to increase and eventually leads to the melting of pennies, making the small change shortage worse.

The SV model is subject to three criticisms. The first is the standard criticism of all cash-in-advance models.[1] It is that the market incompleteness that gives rise to the need for a medium of exchange is simply assumed. It does not arise from fundamentals such as preferences or technologies.

The second, which is closely related, is that even though coins exist in their model, these coins are perfectly divisible. Therefore, the types of problems caused by the indivisibility of coins mentioned in the Ruding quote above, do not arise in their model.

The third is that there is no rationale for the existence of the large coins. In fact, if dollars did not exist, then small change shortages could not arise because the budget constraint and the PIA constraint would be the same. However, there did seem to be a demand for large coins to save on transactions costs for large purchases, particularly those over long distances. This suggests that, in the context of the SV environment, there should also be a "dollar-in-advance" (DIA) constraint. And if the DIA ends up being the more binding of the two, then the implications of SV would be reversed.

The first of these criticisms is addressed by papers which use a random matching, lack of double-coincidence-of-wants framework to motivate an essential role for the use of media of exchange. There are also models which take the indivisibility problem seriously, such as, Zhu (2003). They begin from the historical observations of a shortage of currency and the economists skepticism that shortages of money could exist in a flexible price economy. They use a random matching model with indivisible money and a unit upper bound on money holdings to model a shortage of currency. In their model the shortage arises because there are two groups of agents (say two countries) with differing disutilities of production. There are equilibria in which the agents in one country will export their money and then live in autarky.

Our model differs from theirs in several ways. We have ex ante identical agents (in the equilibria we study, agents hold different portfolios). Also, building on the work of Lee, Wallace, and Zhu (2005)we allow agents to hold multiple coins. Finally, we capture the historical reality that for most of the last millennium there were bimetallic monetary systems in the West by allowing for the existence of two different monies. We generate a demand for large value coins by introducing a cost to carrying coins that is monotonically increasing (we will assume linear) in the number of coins that an agent carries. This allows us to address the third criticism of the SV approach.

The model that we build has another attractive feature that fits well with the Ruding quotation above. In the model, a shortage of low value coins will affect both buyers and sellers. That is, in our model, the terms of trade between buyers and sellers and, in some cases, the ability to trade will depend upon the coin portfolios of both, not just the portfolio of buyers as in the SV analysis.

Once we have built the model, the purpose of the paper will be to examine when small change shortages occur by examining the welfare effects of increasing the quantity of the low value money holding the quantity of high value money constant.

The paper proceeds as follows. In section 2, we build a model with fixed supplies of

[1]see Wallace

metals and coins and examine the implications for small change shortages. In section 3, we modify the model to allow coins to be minted and melted, so that the supplies of coins become endogenous. The final section concludes.

2 Fixed supplies of metals and coins

2.1 Environment

The model has discrete time and an infinite number of periods. There are $N \geq 3$ nonstorable and perfectly divisible commodities.

In addition, there are two metals – silver and gold – that can be turned into coins. There are m_s ounces of silver and m_g ounces of gold in existence in the economy. Each ounces of these commodities gives off one units of a general consumption good at the beginning of each period. [2]

The monetary authority in this environment chooses how many units of each of these commodities to put into a coin. We let b_s be the units of silver than it put in a silver coin and b_g be the number of units of gold that it put in a gold coin. Thus, a silver coin yield a dividend of b_s units of the general consumption good per coin to the holder at the beginning of a period, and a gold coin yields a dividend of b_g units of the general consumption good per coin to the holder at the beginning of a period. The total supplies of the two types of coins are $M_s = m_s/b_s$ and $M_g = m_g/b_g$, respectively.

These gold and silver coins do not have denominations, as was the case with coins throughout most of the time during which commodity monies were used. They are simply amounts of the two metals that have been turned into coins with some type of standardized markings that allow one type of coin to be easily differentiated from a different type of coin. To capture the fact that historically silver coins were less valuable than gold coins, we assume that for technological reasons $b_s < b_g$, silver coins must be less valuable than gold coins in the sense of yielding a lower dividend per coin.[3]

We assume that S is the upper bound on the holdings of silver coins and G is the upper bound of the holdings of gold coins by any agent. Thus, an agent can hold any number of silver coins in the set

$$\mathbf{S} = \{0, 1, 2, ..., S\}.$$

and any number of gold coins in the set

$$\mathbf{G} = \{0, 1, 2, ..., G\}$$

Letting s and g be an agent's holdings of silver and gold coins, respectively, then

$$y = \{(s, g) : s \in \mathbf{S}, g \in \mathbf{G}\}$$

is an agent's portfolio. Let $\mathbf{Y} = \mathbf{S} \bigotimes \mathbf{G}$ be set of all possible portfolios.

[2] Instead of viewing silver and gold as metals, they could be viewed as two different kinds of Lucas trees.

[3] It is not critical to the analysis that the coins be of different metals. Both coins could be gold or both could be silver. What is important is that the two coins have different intrinsic values (different b_j).

There is a $[0,1]$ continuum of infinitely-lived agents in the model. These agents are of N types, and there is an equal proportion of each type. An agent of type n can produce only good n and gets utility from good $n+1$. These preferences are

$$u(q_{n+1}) - q_n$$

with $u(0)=0$, $u'>0$, $u''<0$, and $u'(0)=\infty$. The disutility of production is assumed to be linear without loss of generality. An agent also suffers a utility cost γ for each coin that he holds coming into a period.[4]

In each period agents are matched randomly. There are two types of matches: no coincidence matches and single coincidence matches. Our assumption on agent types rules out double coincidence matches, and therefore, gives rise to the essentiality of a medium of exchange.

We assume that in any match the type and coin portfolio of both agents is known. However, past trading histories are private information and agents are anonymous. These assumptions rule out gift-giving equilibria and the use of credit. Thus, trading can only occur through the use of media of exchange, which is the role that the gold and silver coins can play.

The model's results will be driven by the tension between b_g, b_s, and γ. Given that gold coins have a higher intrinsic value than silver coins, agents would prefer to hold gold coins and trade with silver coins. However, carrying a portfolio so that all trading can be done entirely with silver coins may be more costly than carrying a portfolio such that some potential trades are carried out using some gold coins. That is, agents will economize on the number of silver coins in their portfolios. In addition, this economization on silver coins means that some offers by buyers could include gold coins and could be for goods plus some silver coins. That is, some offers could require the seller to make change. If the seller did not have enough silver coins to make the change demanded by the buyer, then possibly some exchanges might not be made that could be made if the seller was able to make change.

2.2 Consumer choices

We assume that in a single coincidence pairwise meeting, the potential consumer gets to make a take-it-or-leave-it (TIOLI) offer to the potential seller. This offer will be the triple (q, p_s, p_g) where $q \in \Re_+$ is the quantity of production demanded, $p_s \in \mathbb{Z}$ is the net quantity of silver coins offered, and $p_g \in \mathbb{N}$ is the net quantity of gold coins offered. Offers with $p_s < 0$ require the seller to make change.

The set of feasible offers of coins by an agent is

$$\Gamma'(y, \tilde{y}) = \{(p_s, p_g) : -\tilde{s} \le p_s \le \min(s, S-\tilde{s}) \; -\tilde{g} \le p_g \le \min(g, G-\tilde{g})$$

where $\tilde{y}, \tilde{s}$, and $\tilde{g}$ denote the values for the seller. The terms on the left hand side of the inequalities state that feasible offers cannot have the seller giving more coins that he has. The min terms on the righthand side state that feasible offers cannot have the buyer giving

[4]We could assume that coins have different costs of being held, but this would only complicate the analysis without fundamentally changing the results.

more coins than he has or offering so many coins that the upper bound on the seller's coin holdings would be exceeded.

Let $w(s,g) : \mathbf{S} \bigotimes \mathbf{G} \rightarrow \Re_{+}$ be the expected value of an agent's beginning of period portfolio. Then the set of optimal TIOLI offers is

$$\begin{aligned}\Gamma(y,\tilde{y},w) &= \{\sigma : q \in \Re_{+}, (p_s,p_g) \in \Gamma'(y,\tilde{y}), \\ &\quad -q + \beta w(\tilde{s}+p_s, \tilde{g}+p_g) \geq \beta w(\tilde{s},\tilde{g})\}\end{aligned}$$

where the arguments of σ are (q,p_s,p_g). An optimal offer is a feasible offer that satisfies the condition that the seller be no worse off than not trading.

The payoff to the buyer with portfolio y in a trade with a seller with portfolio $\tilde{y}$ is

$$f(s,g,\tilde{s},\tilde{g},w) = \max_{(q,p_s,p_g)\in\Gamma(y,\tilde{y},w)} [u(q) + \beta w(s-p_s, g-p_g)]$$

The payoff to seller in a trade is $\beta w(\tilde{s},\tilde{g})$. The seller is no better or worse off because the buyer's offer extracts all of the surplus in the trade from the seller. Define $\tilde{\Gamma}$ to be the set of optimizers of f.

2.3 Equilibrium

We will consider only steady state equilibria in this section. There are three components needed. The value functions (Bellman equations), the asset transition equations, and the market clearing conditions. We proceed to describe each in turn.

Value functions

The steady state value functions are:

$$w(s,g) = gb_g + sb_s + \frac{1}{N}\sum_{\tilde{s},\tilde{g}} \pi(\tilde{s},\tilde{g}) f(s,g,\tilde{s},\tilde{g},w) + \frac{N-1}{N}\beta w(s,g) - \gamma(s+g)$$

where $\pi(\tilde{s},\tilde{g})$ is the fraction of agents holding $\tilde{s}$ and $\tilde{g}$. The first two terms on the righthand side are returns from holding silver and gold coins, respectively. The third term is the expected payoff from being a buyer in a single coincidence meeting, which occurs with probability $\frac{1}{N}$. The fourth term is the expected payoff from being either the seller in a single coincidence meeting or from being in a nocoincidence meeting. The final term is the cost of carrying the portfolio of coins.

Asset transitions

Define $\lambda(z,z';y,\tilde{y},w)$ to be the probability that a buyer with portfolio y in a meeting with a seller with portfolio $\tilde{y}$ leaves with the meeting with z silver coins and z' gold coins. That is,

$$\lambda(k,k';y,\tilde{y},w) = \begin{cases} 1 & \text{if } k = s - p_s(y,\tilde{y},w) \text{ and } k' = g - p_g(y,\tilde{y},w) \\ 0 & \text{otherwise} \end{cases}$$

Then the asset transitions are

$$\pi_{t+1}(k,k') = \frac{N-2}{N}\pi_t(k,k') + \frac{1}{N}\left\{\sum_{s,\tilde{s},g,\tilde{g}} \pi_t(s,g)\pi_t(\tilde{s},\tilde{g})[\lambda(k,k';\cdot) + \lambda(s+\tilde{s}-k, g+\tilde{g}-k';\cdot)]\right\}$$

The first term on the righthand side is the probability that a meeting is a nocoincidence meeting, in which case no coins change hands. The second term is the fraction of single coincidence meetings in which the buyer leaves with k silver coins and k' gold coins. The third term is the fraction of such meetings in which the seller leaves k silver coins and k' gold coins.

Of course, the asset transition equations must also satisfy

$$\sum_{s\in\mathbf{S}}\sum_{g\in\mathbf{G}} \pi(s,g) = 1$$

Market clearing

The market clearing conditions are that the stocks of gold and silver coins must be held. Let M_g and M_s be the stocks of gold and silver coins, respectively. Then the market clearing conditions are

$$\sum_{s\in\mathbf{S}}\sum_{g\in\mathbf{G}} s\pi(s,g) = M_s$$

$$\sum_{s\in\mathbf{S}}\sum_{g\in\mathbf{G}} g\pi(s,g) = M_g$$

If we think of this economy as a closed economy, these market clearing conditions imply that the stocks of metallic silver and gold used to make coins are $m_s = b_s M_s$ and $m_g = b_g M_g$.

Definition 1 *A steady state equilibrium is a $(w, \pi, \tilde{\Gamma})$ that satisfies the value functions, the asset transition equations, and market clearing.*

Proposition 1 *There exists $(\hat{b_s}, \hat{b_g})$ such that for every $b_s \geq \hat{b_s}$ and $b_g \geq \hat{b_g}$ there exists a steady state equilibrium in which $p_s = p_g = 0$ and this equilibrium is unique.*

The proposition states that if the rates of return to holding gold and silver coins are high enough, then the no trade equilibrium is the only equilibrium that exists.

At this point, we do not have a proof of the proposition, but it seems intuitively obvious.

2.4 Types of equilibria

We have been unable to prove the existence of other equilibria. However, to get some idea of the types of equilibria that could exist, we have computed equilibria for a numerical example. Specifically, we assume $u(q) = \sqrt{(q)}, G = 1, S = 1, M_g = M_s = 1$, and $b_s = 0$.

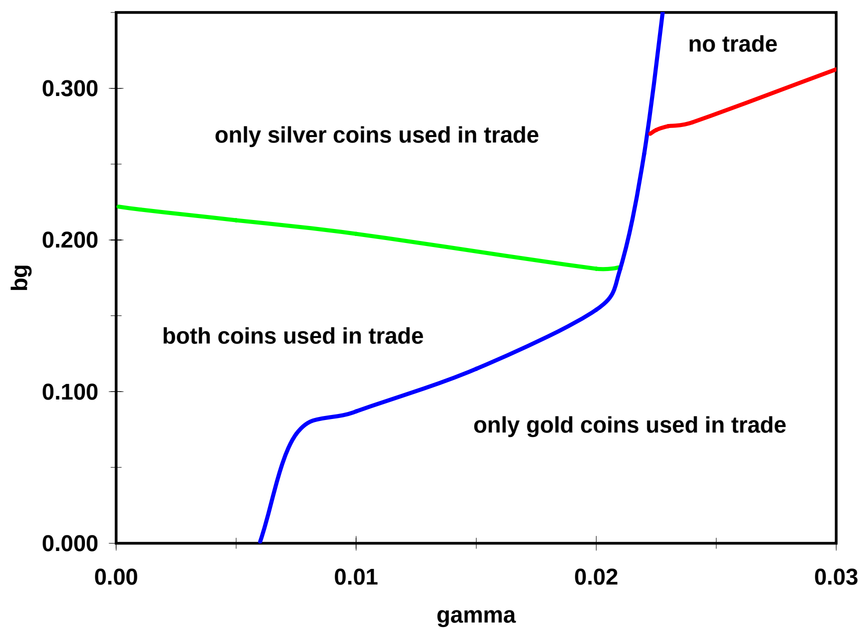

Figure 1: Equilibria for various b_g and γ

The equilibria for this example for various values of b_g and γ are shown in Figure 1. We find that there are four regions:

1. For small b_g and γ both coins circulate in trade.

2. For small γ and large b_g only silver coins are used in trade. The high intrinsic value of the gold coins is such that agents prefer to hold them rather than to trade them for goods. That is, gold coins are worth more in the form of gold than they are as coins for trade. This can be thought of as a case in which Gresham's Law holds.

3. For small b_g and large γ only gold coins are used in trade. When gold coins have little intrinsic value, agents are willing to use them for trade. Hence, the value of silver coins in trade diminishes and at some point becomes less than the cost of holding the silver coins. When this occurs, agents prefer to get rid of the costly silver coins and hold and trade only gold coins. This can be thought of as a case in which a reverse Gresham's Law holds: good coins drive out bad.

4. For large b_g and γ, there is no trade. This is similar to previous case. in that gents prefer to get rid of their silver coins. The difference is that now the intrinsic return on

gold coins is so high that agents prefer to hold on to their gold coins rather than use them for trade. This is the case of Proposition 1.

2.5 Optimal coin sizes

We now consider the question of whether for given m_s and m_g there are optimal sizes for silver and gold coins given the technological constraint that $b_s < b_g$. Once again we have to rely on numerical computations. Specifically, we assume $u(q) = \sqrt(q), G = 2, S = 8, m_s = 0.03, m_g = 0.05$. Results for various $M_s(b_s)$, $M_g(b_g)$, and γ are shown in Table 1.

Assume that the cost of carrying coins is given in the sense that it cannot be affected by actions of the monetary authority. The table shows that for a given cost of carrying coins and for a given size gold coin, there is an optimal size of the silver coin for the economy. For example, when the cost of carrying a coin is $\gamma = 0.001$, then the optimal size silver coin is $b_s = 0.01$ (the supply of silver coins is $M_s = 3$). In addition, the table shows that as the cost of carrying a coin increases, it is optimal to increase the size of the silver coin, if the size of the gold coin is held constant. Specifically, when the cost of carrying a coin increases to $\gamma = 0.01$, the optimal size of the silver coin increases to $b_s = 0.015$ (the supply of silver coins falls to $M_s = 2$). And when the cost of carrying a coin increases to $\gamma = 0.015$, the optimal size of the silver coin increases to $b_s = 0.03$ (the supply of silver coins falls to $M_s = 1$).

The table also shows that if the size of both coins also can be changed as the cost of carrying a coin increases, then it is optimal to increase the sizes of both coins. For example, when the cost of carrying a coin is $\gamma = 0.001$, it is optimal, given our grid, to have the silver coin be of size $b_s = 0.01$ and the gold coin be of size $b_g = 0.05$. However, when the cost of carrying a coin is $\gamma = 0.015$, the optimal sizes of the two coins become $b_s = 0.03$ and $b_g = 0.1$, respectively.

The intuition for these results is that it is optimal to offset somewhat the increased cost of carrying a coin by raising the dividend that a coin pays. This logic is the same as that behind the Friedman Rule.

We also examine the case in which the supply of silver doubles from $m_s = 0.03$ to $m_s = 0.06$. Some results are shown in Table 2. Not surprisingly, the results in the table show that welfare increases as the supply of silver increases; there is more silver to provide dividends. The results in the table also show that, in general, it is optimal to increase the size of the silver coin as the supply of silver increases. Thus, it appears that as a country gets richer, it is optimal for it to increase the metallic content of its coinage. Finally, the results in the table show that, depending on the cost of carrying coins, it is also optimal to increase the supply of silver coins.

We also examine whether the increase in the supply of silver had "price level" effects. That is, we examined the quantity of goods that could be obtained by buyers holding various portfolios of the two types of coins. We do two comparisons. First, we compare the quantity of goods that could be obtained for various offers of the two coins holding the size of the two coins constant, but allowing the quantity of silver coins to increase as m_s increased. For the case when $\gamma = 0.001$, $b_s = 0.015$, and $b_g = 0.05$ we find that for 460 of the 644 possible trades (71 percent), the quantity of goods obtained by buyer decreases as the quantity of silver increases.

Table 1: Welfare and types of trades for various coin sizes and carrying costs, $m_s = 0.03$

	$M_g = 0.5 (b_g = 0.1)$				$M_g = 1 (b_g = 0.05)$			
Silver coin supply (M_s)	1	2	3	4	1	2	3	4
Silver coin size (b_s)	0.03	0.015	0.01	0.0075	0.03	0.015	0.01	0.0075
	$\gamma = 0.001$							
Welfare	1.223	1.414	1.463	1.444	1.300	1.454	1.494	1.470
Types of trades								
silver only	483	521	541	556	468	478	489	494
1 gold less change	138	105	90	77	156	151	146	140
1 gold	5	5	4	4	8	7	8	9
2 gold less change	3	0	0	0	3	3	0	0
Total trades	629	631	635	637	635	639	643	643
	$\gamma = 0.01$							
Welfare	1.118	1.191	1.078	0.785	1.150	1.193	1.041	
Types of trades								
silver only	494	523	544	537	474	462	446	
1 gold less change	129	103	86	88	142	166	175	
1 gold	6	6	6	6	8	10	18	
2 gold less change	0	0	0	0	9	0	0	
Total trades	629	632	636	631	633	638	639	
	$\gamma = 0.015$							
Welfare	1.062	1.045			1.076	1.020		
Types of trades								
silver only	495	522			464	440		
1 gold less change	139	180			126	103		
1 gold	6	6			19	15		
2 gold less change	0	0			10	0		
Total trades	627	631			632	635		

Table 2: Welfare and types of trades for various coin sizes, $m_s = 0.06$

	$M_g = 0.5 (b_g = 0.1)$				$M_g = 1 (b_g = 0.05)$			
Silver coin supply(M_s)	2	3	4	5	2	3	4	5
Silver coin size (b_s)	0.03	0.02	0.015	0.012	0.03	0.020	.015	0.012
	$\gamma = 0.001$							
Welfare	1.658	1.764	1.789	1.718	1.692	1.782	1.793	1.764
Types of trades								
silver only	502	532	543	557	472	495	484	494
1 gold less change	128	102	90	76	158	132	152	141
1 gold only	5	4	4	3	8	7	7	8
2 gold less change	0	0	0	0	3	9	0	0
Total trades	635	638	637	636	641	643	643	643
	$\gamma = 0.01$							
Welfare	1.473	1.455	1.325	1.078	1.458	1.433	1.308	0.767
Types of trades								
silver only	514	536	555	564	479	472	474	462
1 gold less change	114	97	78	69	143	161	157	165
1 gold only	5	6	6	6	8	10	12	16
2 gold less change	0	0	0	0	10	0	0	0
Total trades	633	639	639	639	640	643	643	643

Next, we compare the quantity of goods that could be obtained for various offers of the two coins holding the supply of silver coins constant but allowing the size of the silver coin to increase with the increase in the supply of silver. For the case when $\gamma = 0.001$, $b_s = 0.015$, and $b_g = 0.05$ we find that in only 300 of the 644 possible cases, the quantity of goods obtained by the buyer decreases as the quantity of silver increases.

Thus, whether or not the increase in the quantity of silver could be considered inflationary depends upon the reaction of the monetary authorities. If they react by simply using the additional silver to make more coins, then the "price level" increases. However, if they used to additional silver to make larger coins, but keep the number of coins constant, then there are no price level effects on balance. **add how these are like Lee, Wallace, Zhu**

Token silver coins

Sargent and Velde describe what they call the "standard formula" to solve "the big problem of small change." Part of this standard formula was that small coins should be tokens. In

Table 3: Welfare and types of trades for various M_s and b_g when $b_s = 0$, $\gamma = 0.001$

	$M_g = 0.5(b_g = 0.1)$			$M_g = 1(b_g = 0.05)$		
Silver coin supply(M_s)	1	2	3	1	2	3
Welfare	1.006	1.113	1.105	1.091	1.150	1.014
Types of trades						
silver only	504	533	550	455	463	0
1 gold less change	108	91	78	161	160	0
1 gold	5	4	4	11	11	17
1 gold plus change	0	0	0	0	0	163
Total trades	617	628	632	627	634	643

this section, we examine the case in which there is a fixed supply of silver coins, but the coins are merely tokens in the sense that they throw off no dividend.

To make the results for the token silver coins comparable to those for the case in which the silver coins have intrinsic value, we have added $m_s = 0.03$ to each ex ante welfare from our computation. In other words, we have assumed that the agents in the economy continue to receive the dividends from the economy's stock of silver. However, these dividend flows are not associated with a coin.

The results are shown in Table 3. Comparing these results with those in Table 1, we find that the welfare associated with token silver coins is smaller than that when the silver coins are intrinsically valuable. In other words, welfare is improved when the small coins themselves have value that offsets the cost of carrying them around for trade.

Thus, our analysis shows that implementing the standard formula is not welfare maximizing when there are costs to carrying coins. Instead, welfare is improved by having small coins with some intrinsic value.

The results in Table 3 also show that welfare is higher when the large (gold) coin is smaller relative to the token silver coin. The reason is can be seen in the fact that when the gold coin is smaller, it is used in more trades. As a result, fewer agents have to hold a large number of silver coins in order to make the trades they want and this lowers their cost of carrying coins in order to make transactions.

3 Minting and Melting

In this section we change the previous model such that the supplies of the two types of coins are not fixed. To do this we will permit the minting and melting of the two types of coins. As we will show, introducing the possibility of minting and melting changes the implications of the model.

3.1 The mint and the environment

In our model the mint is a mechanism that converts goods produced by an agent into coins. It will mint a silver coin for b_s units of output and a gold coin for b_g units of output.

We also assume that agents are able to melt coins. When they melt a coin, they receive the amount of output they had to produce to get the coin less any seigniorage that the mint collected and less any costs that they had to incur to melt the coin. We assume that these costs are c_s per silver coin and c_g per gold coin. The result is that melting a silver coin yields $b_s - c_s$ units of output and melting a gold coin yields $b_g - c_g$ units of output. We assume that $b_g - c_g > b_s - c_s$, so that gold coins remain more intrinsically valuable than silver coins.

An agent must pay a fixed cost η to go to the mint or to melt coins. We further assume that at any one time an agent can either only mint or only melt coins. An agent cannot melt one coin and mint the other. We introduce this assumption as a way of incorporating the fact that real world mints only gave back coins of the same metal that an agent brought in. Mints did not permit agents the arbitrage opportunity of bringing in silver and obtaining gold coins or of bringing in gold and obtaining silver coins.

We continue to assume that S is the upper bound on the holdings of silver coins and that G is the upper bound of the holdings of gold coins by any agent. Further, there is still a holding cost of γ per coin.

3.2 Consumer choices

We also make a change to what occurs within a period in the model. Now a period consists of two subperiods. In the first subperiod, agents are randomly matched and can potentially trade as was the case in the fixed supplies of metal model of the previous section. In the second subperiod agents are able to either go to the mint or are able to melt coins.

We begin by considering the payoff function in the second subperiod. Define $z_s \in \mathbb{Z}$ to be quantity of silver coins minted ($z_s < 0$) or melted ($z_s > 0$) and $z_g \in \mathbb{Z}$ to be quantity of gold coins minted ($z_g < 0$) or melted ($z_g > 0$). Let $\phi(z_s, z_g)$ be the utility cost or benefit of going to the mint and minting or melting z_s silver coins and z_g gold coins. Then

$$\phi(z_s, z_g) = \begin{cases} -(b_s z_s + b_g z_g) & \text{if } z_s, z_g \leq 0 \\ \\ +u[(b_s - c_s)z_s + (b_g - c_g)z_g] & \text{if } z_s, z_g \geq 0 \end{cases}$$

Define $v(s, g, w)$ to be the payoff from minting or melting when the agent is holding s silver coins and g coins at the beginning of the second subperiod. Then

$$v(s, g, w) = \max\{\max_{(z_s, z_g) \in \Omega(s,g)} \beta w(s - z_s, g - z_g) + \phi(z_s, z_g) - \eta, 0\}$$

where $\Omega(s, g) = \{(z_s, z_g) | -s \leq z_s \leq S - s, -g \leq z_g \leq G - g\}$.

Given the change to two subperiods, we must also change the payoff function for single coincidence meetings between agents in the first subperiod. It becomes

$$f(s, g, \tilde{s}, \tilde{g}, w) = \max_{(q, p_s, p_g) \in \Gamma(y, \tilde{y}, w)} [u(q) + v(s - p_s, g - p_g)]$$

where the set of optimal TIOLI offers are

$$\begin{aligned}\Gamma(y,\tilde{y},w) &= \{\sigma : q \in \Re_+, (p_s,p_g) \in \Gamma'(y,\tilde{y}),\\ &\quad -q + v(\tilde{s}+p_s, \tilde{g}+p_g) \geq v(\tilde{s},\tilde{g})\}\end{aligned}$$

3.3 Equilibria

There are two components needed to obtain equilibria in this analysis: the value functions (Bellman equations) and the asset transition equations. Market clearing conditions are not needed as the stocks of gold and silver coins are endogenous. Again, we only consider steady states.

3.4 Value functions

Given the payoff functions, the value function at the beginning of the first subperiod is

$$w(s,g) = \frac{1}{N}\sum_{\tilde{s},\tilde{g}} \pi(\tilde{s},\tilde{g}) f(s,g,\tilde{s},\tilde{g},w) + \frac{N-1}{N} v(s,g) - \gamma(s+g)$$

Substituting

$$w(s,g) = \frac{1}{N}\sum_{\tilde{s},\tilde{g}} \pi(\tilde{s},\tilde{g}) \max_{(q,p_s,p_g)\in\Gamma(y,\tilde{y},w)} [u(q) + v(s-p_s, g-p_g)] + \frac{N-1}{N} v(s,g) - \gamma(s+g)$$

3.5 Asset transitions

Define $\lambda(k,k';y,\tilde{y},w)$ as before. Then we can define the asset transition after trading but before agents can go to the mint as

$$\pi_{t+1,1}(k,k') = \frac{1}{N}\left\{\sum_{s,\tilde{s},g,\tilde{g}} \pi_t(g,s)\pi_t(\tilde{s},\tilde{g})[\lambda(k,k';\cdot) + \lambda(s+\tilde{s}-k, g+\tilde{g}-k';\cdot]\right\} + \frac{N-2}{N}\pi_t(k,k')$$

Next, define $\delta(k,k';y'')$ to be the probability that an agent with y'' after trading leaves the mint with k silver coins and k' gold coins. That is

$$\delta(k,k';y'') = \begin{cases} 1 & \text{if } s''-z_s = k \text{ and } g''-z_g = k' \\ \\ 0 & \text{otherwise} \end{cases}$$

Then we can define the asset transitions to be

$$\pi_{t+1}(k,k') = \sum_{s'',g''} \pi_{t+1,1}(s'',g'')\delta(k,k';y'')$$

The asset transition equations must still satisfy

$$\sum_{s\in\mathbf{S}}\sum_{g\in\mathbf{G}} \pi_t(s,g) = 1$$

3.6 Equilibrium

Definition 2 *Given (b_s, c_s, b_g, c_g), an equilibrium is a $(w, \pi, \tilde{\Gamma}, z_s, z_g)$ that satisfies the value functions and the asset transition equations.*

3.7 Results

[yet to be written]

4 Conclusion

In this paper, we build a pairwise matching model with two indivisible commodity monies. Because the monies are indivisible, the model takes seriously the idea that these monies are coins. These commodity monies have different intrinsic values, and agents may hold more than one unit of one or both. We find equilibria in which both monies are use for trades and in which the lower intrinsic-valued money is used to make change in some transactions. We also find that as the higher intrinsic-valued money becomes more valuable, some trades do not occur that occurred previously.

Bibliography

Ales, L., F. Carapella, P. Maziero, and W. E. Weber (????): "A Model of Banknote Discounts," *Journal of Economic Theory*, Forthcoming.

Burdett, K., A. Trejos, and R. Wright (2001): "Cigarette Money," *Journal of Economic Theory*, 99(1-2), 117–142.

Cipolla, C. M. (1956): *Money, Prices, and Civilization in the Mediterranean World, Fifth to Seventeenth Century*. Gordian Press, New York.

Lee, M., N. Wallace, and T. Zhu (2005): "Modeling Denomination Structures," *Econometrica*, 73(3), 949–960.

Rolnick, A. J., F. R. Velde, and W. E. Weber (1996): "The Debasement Puzzle: An Essay on Medieval Monetary History," *The Journal of Economic History*, 56(4), 789–808.

Ruding, R., and J. Y. Akerman (1840): *Annals of the Coinage of Great Britain and its Dependencies*. Printed for J. Hearne, London, 3d , edn.

Sargent, T. J., and F. R. Velde (2002): *The Big Problem of Small Change*, Princeton economic history of the Western world. Princeton University Press, Princeton, NJ.

Spufford, P. (1970): *Monetary Problems and Policies in the Burgundian Netherlands, 1433-1496*. Brill, Leiden.

Wallace, N., and R. Zhou (1997): "A Model of a Currency Shortage," *Journal of Monetary Economics*, 40(3), 555–572.

Wallace, N., and T. Zhu (2007): "Float on a Note," *Journal of Monetary Economics*, 54(2), 229–246.

Zhu, T. (2003): "Existence of a Monetary Steady State in a Matching Model: Indivisible Money," *Journal of Economic Theory*, 112(2), 307–324.