Money, Capital and Collateral

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Abstract

We consider a version of the neoclassical growth model in which some form of intertemporal trade is desirable for agents. If agents are anonymous and lack commitment, the only competitive equilibrium is autarky. We generalize the environment by supposing that agents can commit to repay any debt up to a value related to the value of their physical capital stock. In this sense, commitment is limited, rather than lacking entirely, and capital has value beyond its use in production in that it serves as collateral. For parameters in which collateral constraints do not bind, money is inessential. However, when collateral constraints bind, the supply of private money is too low; and the introduction of fiat money expands the set of feasible trades. Away from the Friedman rule, both fiat money and private debt are essential; with both assets earning an identical real rate of return.

1 Introduction

One of the classic questions in monetary theory deals with the problem of how to rationalize a positive exchange value for a fiat money object. Phrased somewhat differently, under what circumstances might fiat money be *essential* for improving the allocation of resources in an economy?

For a long time, this question constituted somewhat of a puzzle. In particular, standard economic theory is not well-equipped to explain how an intrinsically useless object might come to have value. One answer to this puzzle is provided by the mechanism design literature; a literature that explains how efficient allocations might be implemented in the presence of frictions attributable to limited commitment and private information. In a dynamic context—one that allows for intertemporal trading opportunities—efficiency can generally be improved upon by conditioning allocations on intrinsically useless information; namely, the individual trading histories of all parties involved. In other words, memory is essential.

But to say that memory is essential is not quite the same thing as saying that fiat money is essential; at least, not if fiat money is defined to be some 'tangible' object exchanged on a *quid-pro-quo* basis (i.e., without the aid of a centralized bookkeeping agency). In this case, an additional friction must be introduced to render fiat money essential; namely the absence of a record-keeping technology (this would be the case, for example, if agents were assumed to be anonymous). As stressed by Kocherlakota (1998), fiat money then is just the physical manifestation of the 'intangible' memory that could otherwise be held in a public-access database.

A hallmark of modern monetary theory is that the frictions that make fiat money essential are modeled explicitly. Early versions of models in this branch of the literature were cast in rather stark environments; in particular, environments with a complete lack of commitment and complete anonymity. While these assumptions proved useful for the purpose at hand, the implied frictions are so severe as to preclude the existence of private debt of any form (in particular, private debt that might compete with fiat money as a means of payment).

In our view, a complete lack of commitment appears inconsistent with the fact that much of what passes for 'money' in any well-developed economy is in the form of private debt. An almost universal property these private debt instruments is that they are ultimately collateralized by some form of physical capital. A prominent example is to be found in the small denomination notes issued by chartered banks in the U.S. free-banking era (1836–63).[1] The book-entry liabilities created by modern banks appear to have a similar property.

From a theoretical standpoint then, it seems desirable to think of model economies where physical capital can be used to collateralize private debt instruments, and where these collateralized debt instruments might potentially serve as a means of payment. The challenge would then be to explain why fiat money remains essential in an economy where payments can potentially be made with private money.

There is, of course, a literature that deals with this question. It is well-known, for example, that money and capital can coexist in an overlapping generations (OLG) model. But coexistence in the OLG relies on a peculiar technological condition; it requires that the rate of return on capital

[1] The banknotes of this era were made redeemable in specie and constituted senior claims against the issuing bank's physical property in the event of bankruptcy.

absent money is lower than the population growth rate.[2] Furthermore, when circumstances are such that money and capital do coexist, there is little sense in which capital (or claims collateralized by capital) serve as a payment instrument; rather, capital is valued only for its store-of-value property.

Lagos and Wright (2005) and Rocheteau and Wright (2005) have developed a class of 'search-based' monetary models that manage to avoid the peculiarities associated with the OLG structure, while retaining some degree of analytical tractability.[3] This framework has recently been extended to include physical capital along the lines of a standard neoclassical growth model; see Aruoba and Wright (2003), Lagos and Rocheteau (2004), and Aruoba, Waller and Wright (2006). But in all of these formulations, claims against physical capital in a particular market are precluded from serving as a payment instrument in another (so that payments in this other market must be made in fiat). In effect, these models assume that physical capital cannot be used as collateral to back private debt instruments which might otherwise serve as payment instruments.[4] In light of the evidence to the contrary, such a restriction appears too severe.

The object of our paper is to relax the assumption that physical capital cannot serve as collateral, and then examine the circumstances under which fiat money might nevertheless remain essential. In this version of our paper, we employ a rather crude (but simple) assumption; namely, that agents can commit to honor any debt they issue only up to some value related to the value of their accumulated stock of physical capital.[5] The force of this assumption is to render our environment one in which commitment is *limited*, rather than lacking altogether. One attractive property of our model is that it nests the two extreme cases of no commitment and perfect commitment.

There are, of course, parameter values for which the collateral constraints either bind or remain slack in the absence of money (or memory, if a record-keeping technology is available). When constraints are slack, money (or memory) is inessential. One way to interpret this result is that agents are sufficiently well collateralized, their personal credit histories are irrelevant from the perspective of potential creditors; all that matters is the value of their collateral.

Conversely, when collateral constraints bind in the absence of money, then fiat money expands the set of feasible trades. Away from the Friedman rule, both private debt instruments and fiat money are essential; with both assets earning an identical real rate of return (this is not a model equipped to explain any rate-of-return dominance puzzle). Hence, when collateral constraints bind, there is an insufficient supply of private money. The collateral constraints become less important for lower rates of inflation.

[2]To put this another way, some form of gift-giving (intergenerational transfers) is desirable even with perfect commitment and perfect information. Expressed in yet another way, the allocation that would result in an Arrow-Debreu market is not Pareto optimal (a result that has nothing to do with finitely-lived agents).

[3]While this class of model was originally cast in a search framework, the presence of search frictions has nothing to do with the essentiality of fiat money. The key frictions are as explained earlier: a lack of commitment and anonymity (both properties that naturally stem from a search environment, but can exist even absent search frictions).

[4]Lagos and Rocheteau (2004) allow for two forms of capital, one of which can be used as direct payment. If the payment capital has poor return qualities, then in the absence of money, it will be overaccumulated. The introduction of money then allows people to economize on payment capital. However, this assumes that claims to the non-payment capital cannot be collateralized in any way. Absent this restriction, claims to non-payment capital would drive payment capital and money out of circulation.

[5]A more desirable approach would be to state explicitly the technology available to punish those who default on their obligations. In doing so, one could explicitly write down a set of sequential individual-rationality constraints that induce an endogenous debt limit. We plan to pursue this line of enquiry later, but we suspect that the qualitative results we derive here will remain intact.

Our paper is closely related to Kiyotaki and Moore (2002) who, like us, emphasize the role played by limited commitment (and not search frictions) as the central ingredient in any monetary model. These authors assume that debtors can potentially pledge as collateral some fraction of their future earnings; for us, this collateral instead takes the form of physical capital. Another closely related paper is Ferraris and Watanabe (2007), who also stress the role of physical capital as collateral for private debt. These authors assume that there is an agency (interpreted as a bank) that has the power to seize physical capital. This bank, however, is restricted to issuing loans in fiat money; money loans that must be collateralized with physical capital. Absent such a restriction, it is not immediately clear in their setup why loans in the form of banknotes collateralized by capital would not drive fiat money out of circulation. Nevertheless, many of their main conclusions mirror our own.

We develop our idea in two steps. We first consider a model in which capital is fixed in supply and then extend the model to allow for capital accumulation. The first version of the model allows for simple analytics; and the basic intuition developed there extends to the model that endogenizes capital (this latter version having a more 'neoclassical' flavor).

2 The Basic Environment

The economy consists of a unit mass of *ex ante* identical agents with preferences defined over consumption and work effort $\{c(t), e(t) : t \geq 0\}$. Let preferences be represented by:

$$\sum_{t=0}^{\infty} \beta^t \left[u(c(t)) - g(e(t)) \right];$$

where $u'' < 0 < u', g'$ and $0 \leq g''$, $0 < \beta < 1$.

Each period is divided into two subperiods labeled 'day' and 'night' (there is no discounting between subperiods). Agents are evenly divided among one of two types $i = 1, 2$. Type-1 agents consume during the day and work at night; type-2 agents produce during the day and consume at night. A type i agent is endowed with a type-specific capital k_i and produces output $y_i(t) = k_i + e_i(t)$. Capital cannot be augmented; nor does it depreciate over time.

Let $\theta \geq 0$ denote a Pareto weight. An efficient allocation maximizes:

$$W = [u(c_1) - g(e_1)] + \theta [u(c_2) - g(e_2)]$$

subject to the resource constraints:

$$k_2 + e_2 \geq c_1; \tag{1}$$
$$k_1 + e_1 \geq c_2; \tag{2}$$

and the non-negativity constraints $c_i, e_i \geq 0$. At an interior, the efficient allocation (e_1^*, e_2^*) satisfies:

$$\begin{aligned} \theta u'(k_1 + e_1^*) &= g'(e_1^*); \\ u'(k_2 + e_2^*) &= \theta g'(e_2^*). \end{aligned}$$

2.1 A Parametric Example

Let $u(c) = \log(c)$ and $g(e) = \alpha e$. Then we have:

$$
\begin{aligned}
e_1^* &= \frac{\theta}{\alpha} - k_1; \\
e_2^* &= \frac{1}{\alpha\theta} - k_2;
\end{aligned}
$$

assuming parameters such that $e_i > 0$. Consumption is given by:

$$
\begin{aligned}
c_1^* &= \frac{1}{\alpha\theta}; \\
c_2^* &= \frac{\theta}{\alpha}.
\end{aligned}
$$

Ex post welfare is given by:

$$
\begin{aligned}
W_1^*(\theta) &= -\log\alpha - \log\theta - \theta + \alpha k_1; \\
W_2^*(\theta) &= \log\theta - \log\alpha - \frac{1}{\theta} + \alpha k_2.
\end{aligned}
$$

Note that an equal Pareto weight, $\theta = 1$, maximizes the *ex ante* utility of an agent. However, the agent who ends up with more capital is better off *ex post*.

3 Competitive Equilibrium

Assume that agents cannot trade *ex ante* (i.e., before they know their type). Once type is revealed, however, they have access to a sequence of competitive spot markets. Normalize the price of the day good to unity and let q denote the price of the night good (measured in units of the day good).

The choice problem for the type-1 and type-2 agents are given, respectively, by:

$$
\begin{aligned}
&\max_{e_1} u\left(q(k_1 + e_1)\right) - g(e_1); \\
&\max_{e_2} u\left(q^{-1}(k_2 + e_2)\right) - g(e_2).
\end{aligned}
$$

The associated FOCs are given by:

$$
\begin{aligned}
qu'\left(q(k_1 + e_1)\right) &= g'(e_1); \\
q^{-1}u'\left(q^{-1}(k_2 + e_2)\right) &= g'(e_2);
\end{aligned}
$$

with $c_1 = q(k_1 + e_1)$ and $c_2 = q^{-1}(k_2 + e_2)$. Market-clearing requires:

$$
q(k_1 + e_1) = (k_2 + e_2).
$$

Evaluating the FOCs above at the equilibrium, we have:

$$
\begin{aligned}
qu'\left(k_2 + e_2\right) &= g'(e_1); \\
u'\left(k_1 + e_1\right) &= qg'(e_2);
\end{aligned}
$$

where,

$$q = \frac{(k_2 + e_2)}{(k_1 + e_1)}.$$

Observe that the competitive equilibrium allocation corresponds to a Pareto optimal allocation for Pareto weight $\theta = 1/q$.

For our parametric example, we get

$$\begin{aligned} c_1 &= \frac{1}{\alpha} \\ c_2 &= \frac{1}{\alpha} \\ e_1 &= \frac{1}{\alpha} - k_1 \\ e_2 &= \frac{1}{\alpha} - k_2. \end{aligned}$$

Thus, the Pareto weight θ that corresponds to the *ex-post* competitive equilibrium is equal to 1.

4 Lack of Commitment and Anonymity

Assume now that agents cannot commit to deliver effort. If agents are identifiable, they may nevertheless be threatened with some form of punishment for noncompliance (e.g., banishment from all future trade). For logarithmic preferences, the pain of exclusion would be unbearable so that it would still be feasible to issue claims against effort. To prevent such punishments, assume that agents are anonymous. In particular, assume that agents can hide themselves *and* their capital (so that a punishment in the form of seizing capital is prohibited).

Despite the assumed anonymity and lack of commitment, credit instruments collateralized by the service-flow of capital will nevertheless be valued. The reason for this is simple: the owner-operator of capital bears no cost in producing output with his capital. Note too that as the capital is specific to the owner-operator, it has zero market value.

An agent type-1 begins the period with some claims $b_1 \geq 0$ on type-2 output. In the day-market, he can purchase output by redeeming these claims and also by issuing IOUs $d_1 \geq 0$ at price ϕ_D. These obligations are fully collateralized by his own capital service flow at night. In the night-market, he decides how much to work and exchanges his output for his own IOUs d_1 and claims on future type-2 output, b_1', at price ϕ_N. The problem of a type-1 agent is

$$V_1(b_1) = \max_{c_1, e_1, d_1, b_1'} u(c_1) - g(e_1) + \beta V_1(b_1')$$

subject to

$$\begin{aligned} c_1 &= b_1 + \phi_D d_1 \\ k_1 + e_1 &= d_1 + \phi_N b_1' \\ d_1 &\leq k_1. \end{aligned}$$

An agent type-2 begins the period with some obligations $d_2 \geq 0$. In the day-market, he works e_2 and exchanges his output for his own IOUs and some claims b_2 on type-1 output. In the night-market he redeems d_2 and issues IOUs d_2' backed by his capital service flow to purchase type-1 output. The problem of a type-2 agent is

$$V_2(d_2) = \max_{c_2, e_2, b_2, d_2'} u(c_2) - g(e_2) + \beta V_2(d_2')$$

subject to

$$\begin{aligned} k_2 + e_2 &= d_2 + \phi_D b_2 \\ c_2 &= b_2 + \phi_N d_2' \\ d_2' &\leq k_2. \end{aligned}$$

Proposition 1 *If the planner's solution features strictly positive effort for both types of agents, then the inequality constraints bind.*

Proof. Suppose not, i.e., the competitive equilibrium features effort at efficient levels, $e_i = e_i^* > 0$, and debt strictly lower than capital services, $d_i \leq k_i$. Assume first that $d_i < k_i$. Then, since k_i cannot be consumed by the agent, he can strictly increase his welfare by issuing more debt (zero cost) and working less (a strictly positive gain), a contradiction. Assume now that $d_i = k_i$, then the agent would still like to issue more debt to lower his effort, but cannot. Thus, the debt constraint binds. ■

We thus restrict attention to environments in which the planner's solution features strictly positive effort for both agents. In our parametric example, this implies $k_1, k_2 < 1/\alpha$.

Set $d_1 = k_1$ and the choice problem of a type-1 agent can be stated as:

$$V_1(b_1) = \max_{b_1'} u(b_1 + \phi_D k_1) - g(\phi_N b_1') + \beta V_1(b_1');$$

with FOC:

$$-\phi_N g'\left(\phi_N b_1'\right) + \beta u'\left(b_1' + \phi_D' k_1\right) = 0. \tag{3}$$

For type-2 agents, $d_2' = k_2$., and Hence, the choice problem may be stated as:

$$V_2 = \max_{b_2} u(b_2 + \phi_N k_2) - g(d_2 + \phi_D b_2 - k_2) + \beta V_2';$$

with FOC:

$$u'\left(b_2 + \phi_N k_2\right) - \phi_D g'\left(d_2 + \phi_D b_2 - k_2\right) = 0. \tag{4}$$

The market-clearing conditions here are straightforward: [1] the aggregate debt issued by type-1 agents (k_1) must equal the aggregate saving of the type-2 agents (b_2); and [2] the aggregate debt issued by type-2 agents (k_2) must equal the aggregate saving of the type-1 agents (b_1').

We focus on a stationary equilibrium, so that bond prices are constant. Hence, combining (3) and (4) with market-clearing (and stationarity):

$$\begin{aligned}\phi_N g'(\phi_N k_2) &= \beta u'(k_2 + \phi_D k_1);\\ u'(k_1 + \phi_N k_2) &= \phi_D g'(\phi_D k_1).\end{aligned}$$

Let's now consider the parametric example. In this case, the conditions above imply:

$$\begin{aligned}\alpha\phi_N &= \beta (k_2 + \phi_D k_1)^{-1};\\ (k_1 + \phi_N k_2)^{-1} &= \alpha\phi_D.\end{aligned}$$

A simple closed-form is available when $k_2 = 0$; in which case:

$$\begin{aligned}\phi_N &= \beta\\ \phi_D &= \frac{1}{\alpha k_1}.\end{aligned}$$

The resulting allocation is given by:

$$\begin{aligned}e_1 &= 0\\ e_2 &= \frac{1}{\alpha};\\ c_1 &= \frac{1}{\alpha};\\ c_2 &= k_1.\end{aligned}$$

Of course, in the extreme case for which $k_1 = k_2 = 0$, the economy reverts to autarky. Hence, the availability of collateralizable capital facilitates trade, but it does not, in general, lead to a Pareto efficient outcome. The reason for this is because in general, positive levels of work effort are required for efficiency. But given agent anonymity and the inability to commit to future levels of work effort, the equilibrium level of work effort is generally suboptimal from a social perspective.

Note that if $k_1 \geq 1/\alpha$ then $\phi_D \phi_N = \beta$ and we achieve the first-best allocation and there is no role for fiat money.

5 Fiat Money

5.1 Fixed money supply

Assume there is constant stock of fiat money M and that this money is initially endowed evenly among type-1 agents. Let v_D and v_N denote the value of money measured in units of day and night output, respectively. Let m_1 denote the nominal balances held by a type-1 agent at the beginning of the day market. Then a type-1 agent faces the following constraints (invoking $d_1 = k_1$):

$$\begin{aligned}c_1 &= b_1 + v_D m_1 + \phi_D k_1;\\ e_1 &= v_N m_1' + \phi_N b_1'.\end{aligned}$$

That is, the type-1 agent exerts effort to purchase money and bonds in the night market. As both of these financial instruments are risk-free, it must be the case that their returns are equated in equilibrium (if the two assets are to be willingly held by agents); i.e.,

$$\left(\frac{v_N}{v_D'}\right) = \phi_N. \tag{5}$$

Anticipating that this must be the case, define $z_1 \equiv b_1 + v_D m_1$ and rewrite the budget constraints as:

$$\begin{aligned} c_1 &= z_1 + \phi_D k_1; \\ e_1 &= \phi_N z_1'. \end{aligned}$$

The problem of a type-1 agent may therefore be expressed as:

$$V_1(z_1) = \max_{z_1'} u(z_1 + \phi_D k_1) - g(\phi_N z_1') + \beta V_1(z_1').$$

The first-order condition is given by:

$$\phi_N g'\left(\phi_N z_1'\right) = \beta u'\left(z_1' + \phi_D' k_1\right). \tag{6}$$

Note that given the rate-of-return equality condition (5), the composition of desired future wealth z_1' between money and bonds is indeterminate at the individual level. In other words, agents anticipate that payment for goods and services may be made in either fiat or private money.

A type-2 agent faces the following constraints (invoking $d_2' = k_2$):

$$\begin{aligned} c_2 &= b_2 + v_N m_2 + \phi_N k_2; \\ e_2 &= v_D m_2 + \phi_D b_2. \end{aligned}$$

In this case, the relevant no-arbitrage condition is:

$$\left(\frac{v_D}{v_N}\right) = \phi_D. \tag{7}$$

Anticipating that this must be the case, define $z_2 \equiv b_2 + v_N m_2$, so that his budget constraints may be written as:

$$\begin{aligned} c_2 &= z_2 + \phi_N k_2; \\ e_2 &= \phi_D z_2. \end{aligned}$$

The problem of a type-2 agent may therefore be expressed as:

$$V_2 = \max_{z_2} u(z_2 + \phi_N k_2) - g(\phi_D z_2) + \beta V_2'.$$

The first-order condition is given by:

$$u'\left(z_2 + \phi_N k_2\right) = \phi_D g'\left(\phi_D z_2\right). \tag{8}$$

Market-clearing here requires:

$$\begin{aligned} b_1' &= k_2; \\ b_2 &= k_1; \\ m_1' &= 2M; \\ m_2 &= 2M. \end{aligned}$$

We focus again of a stationary equilibrium, so that bond prices and the values of money are constant over time. We can use the FOCs (6) and (8) and the resource constraints (1) and (2) to solve for ϕ_D, ϕ_N, z_1 and z_2:

$$\begin{aligned} \phi_N g'(\phi_N z_1) &= \beta u'(z_1 + \phi_D k_1); \\ \phi_D g'(\phi_D z_2) &= u'(z_2 + \phi_N k_2); \\ z_1 + \phi_D k_1 &= k_2 + \phi_D z_2; \\ z_2 + \phi_N k_2 &= k_1 + \phi_N z_1. \end{aligned}$$

From the last two equations, we can get the prices of bonds as functions of z_1 and z_2:

$$\begin{aligned} \phi_D &= \frac{z_1 - k_2}{z_2 - k_1}; \\ \phi_N &= \frac{z_2 - k_1}{z_1 - k_2}. \end{aligned}$$

Thus, note that $\phi_D \phi_N = 1$.

Using the FOCS (6) and (8) we can solve z_1 and z_2:

$$\begin{aligned} \frac{z_2 - k_1}{z_1 - k_2} g'\Big(\frac{(z_2 - k_1) z_1}{z_1 - k_2}\Big) &= \beta u'\Big(\frac{(z_2 - k_1) z_1 + (z_1 - k_2) k_1}{z_2 - k_1}\Big); \\ \frac{z_1 - k_2}{z_2 - k_1} g'\Big(\frac{(z_1 - k_2) z_2}{k_1 - z_2}\Big) &= u'\Big(\frac{(z_1 - k_2) z_2 + (z_2 - k_1) k_2}{k_2 - z_1}\Big). \end{aligned}$$

Then, we can solve for v_D and v_N using the market clearing conditions for money:

$$\begin{aligned} v_D &= \frac{z_1 - k_2}{2M}; \\ v_N &= \frac{z_2 - k_1}{2M}. \end{aligned}$$

For our parametric example we get

$$\begin{aligned} z_1 &= \frac{1}{\alpha} - \frac{k_1}{\beta}; \\ z_2 &= \beta\Big(\frac{1}{\alpha} - k_2\Big). \end{aligned}$$

The allocation in the monetary equilibrium is

$$\begin{aligned} c_1 &= \frac{1}{\alpha}; \\ c_2 &= \frac{\beta}{\alpha}; \\ e_1 &= \frac{\beta}{\alpha} - k_1; \\ e_2 &= \frac{1}{\alpha} - k_2; \end{aligned}$$

with

$$\begin{aligned} \phi_D &= \frac{1}{\beta}; \\ \phi_N &= \beta; \\ v_D &= \frac{\beta - \alpha(k_1 - \beta k_2)}{2M\alpha\beta}; \\ v_N &= \frac{\beta - \alpha(k_1 - \beta k_2)}{2M\alpha}. \end{aligned}$$

5.2 Growing money supply

Now assume that money grows at a constant rate, i.e.,

$$M' = \mu\ M,$$

where $\mu \geq \beta$.

New money is introduced as a lump-sum transfer τ to type-1 agents at the beginning of each period. Thus, new money is used for transactions during the period. If $\mu < 1$, then the money supply contracts and τ constitutes a lump-sum tax, payable in fiat money. The lump-sum transfer per agent is

$$\tau = (\mu - 1)2M$$

Consumption and effort for an agent type-1 satisfy (invoking $d_1 = k_1$)

$$\begin{aligned} c_1 &= b_1 + v_D(m_1 + \tau) + \phi_D k_1; \\ e_1 &= v_N m_1' + \phi_N b_1'. \end{aligned}$$

Define as before $z_1 \equiv b_1 + v_D m_1$. Then rewrite the above budget constraints as

$$\begin{aligned} c_1 &= z_1 + v_D \tau + \phi_D k_1; \\ e_1 &= \phi_N z_1'. \end{aligned}$$

Other than the value for c_1 the problems of agents type-1 and 2 look the same. Thus we get the following FOCs and resource constraints:

$$\begin{aligned} \phi_N g'(\phi_N z_1') &= \beta u'(z_1' + \phi_D' k_1 + v_D' \tau'); \\ \phi_D g'(\phi_D z_2) &= u'(z_2 + \phi_N k_2) \\ z_1 + v_D \tau + \phi_D k_1 &= k_2 + \phi_D z_2; \\ z_2 + \phi_N k_2 &= k_1 + \phi_N z_1', \end{aligned}$$

where (from the market clearing condition for money)

$$v_D \tau = (\mu - 1)(z_1 - k_2).$$

We focus on a stationary equilibrium. From the resource constraints, we can get the prices of bonds as functions of z_1, z_2 and μ:

$$\begin{aligned} \phi_D &= \mu \frac{z_1 - k_2}{z_2 - k_1}; \\ \phi_N &= \frac{z_2 - k_1}{z_1 - k_2}. \end{aligned}$$

Note that $\phi_D \phi_N = \mu$. We now show optimality of the Friedman rule.

Proposition 2 *If the the planner's solution features strictly positive effort for both types of agents, then the Friedman rule, $\mu = \beta$, implements the first-best allocation with Pareto weight $\theta = \frac{1}{\phi_D} \frac{g'(e_1)}{g'(e_2)} = \frac{\phi_N}{\beta} \frac{g'(e_1)}{g'(e_2)}$.*

Proof. If both effort levels are strictly positive in the planner's solution, then by Proposition 1, there is a role for fiat money since the collateral constraints bind in the absence of money.

At an interior, the solution to the planner's problem is

$$\begin{aligned} u'(k_2 + e_2^*) &= \theta g'(e_2^*) \\ \theta u'(k_1 + e_1^*) &= g'(e_1^*). \end{aligned}$$

These two equations imply

$$u'(k_1 + e_1^*) \; u'(k_2 + e_2^*) = g'(e_1^*) \; g'(e_2^*).$$

A monetary equilibrium solves the following conditions

$$\begin{aligned} u'(k_2 + e_2) &= \frac{\phi_N}{\beta} g'(e_1) \\ u'(k_1 + e_1) &= \phi_D g'(e_2), \end{aligned}$$

which imply

$$u'(k_1 + e_1) \; u'(k_2 + e_2) = \frac{\phi_D \phi_N}{\beta} \; g'(e_1) \; g'(e_2).$$

At the Friedman rule, $\phi_D \phi_N = \beta$ and so the planner's allocation and the monetary equilibrium imply the same condition.

Now we verify that there exists a corresponding θ. From the monetary equilibrium conditions we have

$$u'(k_2 + e_2) = \frac{\phi_N}{\beta} \frac{g'(e_1)}{g'(e_2)} \; g'(e_2).$$

Let $\theta = \frac{\phi_N}{\beta}\frac{g'(e_1)}{g'(e_2)}$ and so the above equation simplifies to

$$u'(k_2 + e_2^*) = \theta g'(e_2^*),$$

i.e., same as the first of the planner's condition.

The second condition from the monetary equilibrium can be written as

$$u'(k_1 + e_1) = \phi_D \frac{g'(e_2)}{g'(e_1)}\ g'(e_1).$$

Since $\phi_D \phi_N = \beta$ we get

$$u'(k_1 + e_1) = \frac{\beta}{\phi_N}\frac{g'(e_2)}{g'(e_1)}\ g'(e_1).$$

Apply the value for θ and we get

$$\theta u'(k_1 + e_1) = g'(e_1),$$

i.e., same as the second of the planner's condition.

Given our assumptions on u and g, the solutions for e_1 and e_2 are unique and identical to the planner's solution. Thus, for $\theta = \frac{\phi_N}{\beta}\frac{g'(e_1)}{g'(e_2)}$, we have $e_1 = e_1^*$ and $e_2 = e_2^*$. ■

For our parametric example we get

$$\begin{aligned} z_1 &= \frac{1}{\alpha\mu} - \frac{k_1}{\beta}; \\ z_2 &= \beta\Big(\frac{1}{\alpha} - k_2\Big). \end{aligned}$$

The allocation in the monetary equilibrium with a constant money growth rate is

$$\begin{aligned} c_1 &= \frac{1}{\alpha} - (\mu - 1)k_2; \\ c_2 &= \frac{\beta}{\alpha\mu(1 - \alpha k_2(\mu - 1))}; \\ e_1 &= \frac{\beta - \alpha\mu k_1(1 - \alpha k_2(\mu - 1))}{\alpha\mu(1 - \alpha k_2(\mu - 1))}; \\ e_2 &= \frac{1}{\alpha} - \mu k_2; \end{aligned}$$

with

$$\begin{aligned} \phi_D &= \frac{\mu(1 - \alpha k_2(\mu - 1))}{\beta}; \\ \phi_N &= \frac{\beta}{1 - \alpha k_2(\mu - 1)}; \end{aligned}$$

Note, that from the variables above, only e_1 depends on k_1. All others[6] depend only on k_2.

The Friedman rule implements the first-best with Pareto weight

$$\theta = \frac{1}{1 + \alpha(1 - \beta)k_2}.$$

Thus, k_2 determines which point of the Pareto frontier is achieved under the Friedman rule.

[6]Except v_D and v_N which depends on both k_1 and k_2.

6 Welfare comparisons

Let us verify our results with our parametric example. In addition, if we set $k_2 = 0$, then it is easy to compare welfare across the different cases we have covered. We will compare the flow utility of a "representative" agent. We restrict attention to environments in which a planner would recommend strictly positive effort to both types of agents and where the competitive equilibrium without money is not autarky. Thus, we focus on the case $k_1 \in (0, 1/\alpha)$. We set $\theta = 1$, which is the Pareto weight that corresponds to the first-best allocation implemented by the Friedman rule when $k_2 = 0$.

The Pareto optimal allocation yields

$$W^{PO} = -1 + \frac{\alpha k_1}{2} - \log \alpha.$$

The flow utility in the competitive equilibrium without money is

$$W^{CE} = \frac{-1 + \log k_1 - \log \alpha}{2}.$$

The difference between the two flow utilities is

$$W^{PO} - W^{CE} = \frac{-1 + \alpha k_1 - \log \alpha k_1}{2},$$

which is strictly positive for any $k_1 \in (0, 1/\alpha)$.

The flow utility in a monetary equilibrium is

$$W^{ME} = -\frac{1 + \frac{\beta}{\mu} - \log(\frac{\beta}{\mu}) - \alpha k_1 + 2 \log \alpha}{2}.$$

Taking the derivative of the above expression with respect to μ yields

$$\frac{dW^{ME}}{d\mu} = \frac{\beta - \mu}{2\mu^2}$$

and so $\mu = \beta$, i.e., the Friedman rule, maximizes welfare for the agent[7].

Under the Friedman rule, a monetary equilibrium exists if $k_1 < 1/\alpha$ (otherwise, v_D and v_N are not strictly positive). As shown above, the Pareto optimal allocation strictly dominates the competitive equilibrium without money for this case. The flow utility in the monetary equilibrium under the Friedman rule is

$$W^{FR} = -1 + \frac{\alpha k_1}{2} - \log \alpha,$$

which is identical to the flow utility of the Pareto optimal allocation.

[7] Taking the second derivative yields $-\frac{2\beta - \mu}{2\mu^3}$, which is negative for $\mu = \beta$, so this is indeed a maximum.

7 Endogenous capital stock

7.1 Planner's problem

Assume now that capital can be accumulated and depreciates at rate δ. The production function, $F(k, e)$, is constant returns to scale and exhibits diminishing returns in both inputs. We begin by analyzing the planner's problem when there are no informational problems.

The resource constraints are

$$\begin{aligned} c_1 + k_2' &= F(k_2, e_2) + (1-\delta)k_2 \\ c_2 + k_1' &= F(k_1, e_1) + (1-\delta)k_1. \end{aligned}$$

We maintain the assumption that capital is specific to the owner-operator. Thus, consumption can only be supplied with production. In other words, net investment has to be positive

$$\begin{aligned} k_1' - (1-\delta)k_1 &\geq 0 \\ k_2' - (1-\delta)k_2 &\geq 0. \end{aligned}$$

For a given Pareto weight $\theta \geq 0$, the planner's problem is

$$W(k_1, k_2) = \max_{k_1', k_2', e_1, e_2} u(c_1) - g(e_1) + \theta\big(u(c_2) - g(e_2)\big) + \beta W(k_1', k_2')$$

subject to the resource constraints and the non-negativity constraints on net investment.

Assuming the non-negativity constraints do not bind, the first-order conditions are

$$\begin{aligned} -u_{c,2} + \beta u_{c,2}'(F_{k,2}' + 1 - \delta) &= 0 \\ -u_{c,1} + \beta u_{c,1}'(F_{k,1}' + 1 - \delta) &= 0 \\ -g_{e,1} + \theta u_{c,2} F_{e,1} &= 0 \\ u_{c,1} F_{e,2} - \theta g_{e,2} &= 0. \end{aligned}$$

If both types of agents start with the same capital, then with $\theta = 1$, all agents receive the same lifetime utility.

In steady state, we get following conditions

$$\begin{aligned} F_{k,1} + 1 - \delta - \frac{1}{\beta} &= 0 \\ F_{k,2} + 1 - \delta - \frac{1}{\beta} &= 0 \\ \theta u_{c,2} F_{e,1} - g_{e,1} &= 0 \\ u_{c,1} F_{e,2} - \theta g_{e,2} &= 0, \end{aligned}$$

where

$$\begin{aligned} c_1 &= F(k_2, e_2) - \delta k_2 \\ c_2 &= F(k_1, e_1) - \delta k_1. \end{aligned}$$

Clearly, the marginal product of capital is equated across agents in steady state. Given our assumptions on F, this implies $k_1/e_1 = k_2/e_2$ and thus $F_{e,1} = F_{e,2}$. Let r^* and w^* be the steady state first-best marginal products of capital and labor, respectively. Then

$$\begin{aligned} r^* + 1 - \delta - \frac{1}{\beta} &= 0 \\ \theta u_{c,2} w^* - g_{e,1} &= 0 \\ u_{c,1} w^* - \theta g_{e,2} &= 0. \end{aligned}$$

7.2 Lack of commitment and anonymity

As in the case of the simple model, we now assume that agents are anonymous and cannot commit to work. Since capital is specific to the owner-operator, there is no market for the purchase or rental of capital. However, we allow the capital stock to be used as collateral. That is, assume that a type-i agent can commit to honor any debt up to, but not beyond the value $H_i(k)$; which is assumed to be increasing in k. For now, we will not be explicit about the particular functional form, but one can see that it may include $F(k, 0)$ and some fraction of the capital stock.

We start by looking at the case without fiat money. Thus, the only means of payment is private debt.

An agent type-1 faces the following budget and debt constraints:

$$\begin{aligned} c_1 &= b_1 + \phi_D d_1 \\ F(k_1, e_1) + (1-\delta)k_1 &= d_1 + \phi_N b_1' + k_1' \\ d_1 &\leq H_1(k_1). \end{aligned}$$

As with the simple model, we will restrict attention to environments in which the inequality constraint on debt binds. Thus, setting $d_1 = H_1(k_1)$, the problem of an agent type-1 is

$$\begin{aligned} V_1(k_1, b_1) = \max_{e_1, b_1'} \; & u(b_1 + \phi_D H_1(k_1)) - g(e_1) \\ & + \beta V_1(F(k_1, e_1) + (1-\delta)k_1 - H_1(k_1) - \phi_N b_1', b_1') \end{aligned}$$

The first-order conditions are

$$\begin{aligned} -g_{e,1} + \beta w_1 V_{k,1}' &= 0 \\ -V_{k,1}' \phi_N + V_{b,1}' &= 0. \end{aligned}$$

The envelope condition implies

$$\begin{aligned} V_{k,1} &= u_{c,1} \phi_D H_{k,1} + \beta V_{k,1}' (r_1 + 1 - \delta - H_{k,1}) \\ V_{b,1} &= u_{c,1} \end{aligned}$$

From the first-order conditions, we have

$$\beta V_{k,1}' = \frac{g_{e,1}}{w_1}.$$

Plug this expression into the equation for $V_{k,1}$ and get

$$V_{k,1} = u_{c,1}\phi_D H_{k,1} + \frac{g_{e,1}}{w_1}(r_1 + 1 - \delta - H_{k,1}).$$

The first-order conditions can now be written as

$$\begin{aligned}
\frac{g_{e,1}}{w_1} &= \beta\Big(u'_{c,1}\phi'_D H'_{k,1} + \frac{g'_{e,1}}{w'_1}(r'_1 + 1 - \delta - H'_{k,1})\Big) \\
\phi_N &= \beta\frac{u'_{c,1}w_1}{g_{e,1}}.
\end{aligned}$$

An agent type-2 faces the following budget and debt constraints

$$\begin{aligned}
c_2 &= b_2 + \phi_N d'_2 \\
F(k_2, e_2) + (1-\delta)k_2 &= d_2 + \phi_D b_2 + k'_2 \\
d'_2 &\leq H_2(k_2).
\end{aligned}$$

As explained above, in equilibrium $d'_2 = H_2(k_2)$ and so the problem of an agent type-2 is

$$\begin{aligned}
V_2(k_2, d_2) = \max_{e_2, b_2}\ & u(b_2 + \phi_N H_2(k_2)) - g(e_2) \\
& + \beta V_2(F(k_2, e_2) + (1-\delta)k_2 - d_2 - \phi_D b_2, H_2(k_2))
\end{aligned}$$

Following similar steps as with the type-1 agent's problem, we get the following first-order conditions

$$\begin{aligned}
\frac{g_{e,2}}{w_2} &= \beta\Big(u'_{c,2}\phi'_N H'_{k,2} + \frac{g'_{e,2}}{w'_2}(r'_2 + 1 - \delta) - \beta\frac{g''_{e,2}}{w''_2}H''_{k,2}\Big) \\
\phi_D &= \frac{u_{c,2}w_2}{g_{e,2}}.
\end{aligned}$$

The market clearing conditions are

$$\begin{aligned}
b'_1 &= H_2(k_2) \\
b_2 &= H_1(k_1).
\end{aligned}$$

7.2.1 Steady state

In steady state, we can use the budget constraints of the agents to get expressions for ϕ_D and ϕ_N

$$\begin{aligned}
\phi_D &= \frac{F(k_2, e_2) - \delta k_2 - H_2(k_2)}{H_1(k_1)} \\
\phi_N &= \frac{F(k_1, e_1) - \delta k_1 - H_1(k_1)}{H_2(k_2)}.
\end{aligned}$$

From the expressions above, it seems that the amount of collateral cannot be too large in order for the price of bonds to be strictly positive. Suppose $H_i = h_i k_i$, for some constant $h_i > 0$; then, for an equilibrium to exist, $h_i < F_i(1, e_i/k_i) - \delta$. Thus, the requirement is not the *amount* of collateral be small enough, but that the *fraction* of capital that can be used as collateral be low enough.

To solve for k_1, k_2, e_1 and e_2 in steady state, we use the first-order conditions from the agents' problems

$$\begin{aligned} \Big(\frac{\phi_D\phi_N}{\beta} - 1\Big)H_{k,1} + r_1 + 1 - \delta - \frac{1}{\beta} &= 0 \\ (\phi_D\phi_N - \beta)H_{k,2} + r_2 + 1 - \delta - \frac{1}{\beta} &= 0 \\ \phi_N g_{e,1} - \beta u_{c,1} w_1 &= 0 \\ \phi_D g_{e,2} - u_{c,2} w_2 &= 0, \end{aligned}$$

where

$$\begin{aligned} c_1 &= (F(k_2, e_2) - \delta k_2) \\ c_2 &= (F(k_1, e_1) - \delta k_1). \end{aligned}$$

The competitive equilibrium is efficient if and only if $\phi_D\phi_N = \beta$. Thus, unless parameter values are such that this condition is met, the competitive equilibrium without fiat money is inefficient.

7.3 Fiat money

As in the case of the simple model, we will use the fact that the returns of money and bonds are equal in equilibrium. Thus,

$$\begin{aligned} \phi_D &= \frac{v_D}{v_N} \\ \phi_N &= \frac{v_N}{v'_D}. \end{aligned}$$

7.3.1 Type-1 agent

An agent type-1 faces the following budget and debt constraints:

$$\begin{aligned} c_1 &= b_1 + \phi_D d_1 + v_D(m_1 + \tau) \\ F(k_1, e_1) + (1-\delta)k_1 &= d_1 + \phi_N b'_1 + k'_1 + v_N m'_1 \\ d_1 &\leq H_1(k_1), \end{aligned}$$

where τ is a lump-sum transfer of money that the agent receives at the beginning of the period.

Let $z_1 \equiv b_1 + v_D m_1$ and assume that the debt constraint is satisfied with equality. Then

$$\begin{aligned} c_1 &= z_1 + \phi_D H_1(k_1) + v_D\tau \\ k'_1 &= F(k_1, e_1) + (1-\delta)k_1 - H_1(k_1) - \phi_N z'_1. \end{aligned}$$

The problem of an agent type-1 is

$$\begin{aligned}V_1(k_1,z_1) = \max_{e_1,z_1'} &\, u(z_1+\phi_D H_1(k_1)+v_D\tau)-g(e_1)\\ &+\beta V_1(F(k_1,e_1)+(1-\delta)k_1-H_1(k_1)-\phi_N z_1', z_1')\end{aligned}$$

The first-order conditions are

$$\begin{aligned}-g_{e,1}+\beta w_1 V_{k,1}' &= 0\\ -V_{k,1}'\phi_N+V_{z,1}' &= 0.\end{aligned}$$

The envelope condition implies

$$\begin{aligned}V_{k,1} &= u_{c,1}\phi_D H_{k,1}+\beta V_{k,1}'(r_1+1-\delta-H_{k,1})\\ V_{b,1} &= u_{c,1}.\end{aligned}$$

From the first-order conditions, we have

$$\beta V_{k,1}' = \frac{g_{e,1}}{w_1}.$$

Using this expression we get

$$V_{k,1} = u_{c,1}\phi_D H_{k,1}+\frac{g_{e,1}}{w_1}(r_1+1-\delta-H_{k,1}).$$

The first-order conditions can now be written as

$$\begin{aligned}\frac{g_{e,1}}{w_1} &= \beta\Big(u_{c,1}'\phi_D' H_{k,1}'+\frac{g_{e,1}'}{w_1'}(r_1'+1-\delta-H_{k,1}')\Big)\\ \phi_N &= \beta\frac{u_{c,1}'w_1}{g_{e,1}}.\end{aligned}$$

7.3.2 Type-2 agent

An agent type-2 faces the following budget and debt constraints:

$$\begin{aligned}c_2 &= b_2+\phi_N d_2'+v_N m_2\\ F(k_2,e_2)+(1-\delta)k_2 &= d_2+\phi_D b_2+k_2'+v_D m_2\\ d_2' &\le H_2(k_2).\end{aligned}$$

Let $z_2 \equiv b_2+v_N m_2$ and assume that the debt constraint is satisfied with equality. Then

$$\begin{aligned}c_2 &= z_2+\phi_N H_2(k_2)\\ k_2' &= F(k_2,e_2)+(1-\delta)k_2-d_2-\phi_D z_2.\end{aligned}$$

The problem of an agent type-2 is

$$\begin{aligned} V_2(k_2, d_2) = \max_{e_2, z_2} \; & u(z_2 + \phi_N H_2(k_2)) - g(e_2) \\ & + \beta V_2(F(k_2, e_2) + (1-\delta)k_2 - d_2 - \phi_D z_2, H_2(k_2)) \end{aligned}$$

Following similar steps as with the type-1 agent's problem, we get the following first-order conditions

$$\begin{aligned} \frac{g_{e,2}}{w_2} &= \beta\Big(u'_{c,2}\phi'_N H'_{k,2} + \frac{g'_{e,2}}{w'_2}(r'_2 + 1 - \delta) - \beta\frac{g''_{e,2}}{w''_2}H''_{k,2}\Big) \\ \phi_D &= \frac{u_{c,2}w_2}{g_{e,2}}. \end{aligned}$$

7.3.3 Monetary equilibrium

The market clearing conditions are

$$\begin{aligned} b'_1 &= H_2(k_2) \\ b_2 &= H_1(k_1) \\ m'_1 &= \mu 2M \\ m_2 &= \mu 2M, \end{aligned}$$

where $\mu \geq \beta$ is the money growth rate.

Using the definitions of z_1 and z_2 we can write the bond market clearing conditions as

$$\begin{aligned} z'_1 - v'_D \mu 2M &= H_2(k_2) \\ H_1(k_1) &= z_2 - v_N \mu 2M, \end{aligned}$$

which imply the following expressions for v'_D and v_N:

$$\begin{aligned} v'_D &= \frac{z'_1 - H_2(k_2)}{\mu 2M} \\ v_N &= \frac{z_2 - H_1(k_1)}{\mu 2M}. \end{aligned}$$

The interpretation of the equations above is that for a monetary equilibrium to exist, it has to be the case that the means of payment that agents want to acquire (z'_1 and z_2) have to be larger that the amount of debt that agents of the other type can issue.

We can use the fact that the beginning-of-period amount of bonds held by type-1 agents is equal to the beginning-of-period debt owed by type-2 agents, i.e., $b_1 = d_2$, to get an expression for v_D. Proceeding as above we get

$$v_D = \frac{z_1 - d_2}{2M}.$$

This implies a gross inflation rate (in terms of day market prices) of

$$\frac{v_D}{v'_D} = \mu \, \frac{z_1 - d_2}{z'_1 - H_2(k_2)}.$$

Given that the returns of bonds and money are equalized in equilibrium, we can derive expressions for ϕ_D and ϕ_N

$$\begin{aligned}\phi_D &= \mu \, \frac{z_1 - d_2}{z_2 - H_1(k_1)} \\ \phi_N &= \frac{z_2 - H_1(k_1)}{z_1' - H_2(k_2)}.\end{aligned}$$

Since $\tau = (\mu - 1)2M$, we can derive the expression for the real value of money transfers

$$v_D \tau = (\mu - 1)(z_1 - d_2).$$

7.3.4 Steady state

In steady state, $d_2 = H_2(k_2)$ and thus the expressions for the prices of bonds reduce to

$$\begin{aligned}\phi_D &= \mu \, \frac{z_1 - H_2(k_2)}{z_2 - H_1(k_1)} \\ \phi_N &= \frac{z_2 - H_1(k_1)}{z_1 - H_2(k_2)}.\end{aligned}$$

Note that $\phi_D \phi_N = \mu$.

From the budget constraints that give the expressions for k_1' and k_2' we have

$$\begin{aligned}z_1 \, \frac{z_2 - H_1(k_1)}{z_1 - H_2(k_2)} &= F(k_1, e_1) - \delta k_1 - H_1(k_1) \\ \mu z_2 \, \frac{z_1 - H_2(k_2)}{z_2 - H_1(k_1)} &= F(k_2, e_2) - \delta k_2 - H_2(k_2).\end{aligned}$$

We can now solve for z_1, z_2, ϕ_D and ϕ_N as functions of k_1, k_2, e_1 and e_2. Define

$$\chi \equiv \frac{F(k_2, e_2) - \delta k_2 + (\mu - 1) H_2(k_2)}{F(k_1, e_1) - \delta k_1}.$$

Then

$$\begin{aligned}z_1 &= \frac{\chi}{\mu} \left(F(k_1, e_1) - \delta k_1 - H_1(k_1) \right) \\ z_2 &= \frac{1}{\chi} \left(F(k_2, e_2) - \delta k_2 - H_2(k_2) \right) \\ \phi_D &= \chi \\ \phi_N &= \frac{\mu}{\chi}.\end{aligned}$$

Note that ϕ_D and ϕ_N depend on $H_2(k_2)$, but not $H_1(k_1)$.

The steady state is characterized by the four first-order conditions from the problems of the agents. Thus, k_1, k_2, e_1 and e_2 solve

$$\begin{aligned}
\Big(\frac{\mu}{\beta} - 1\Big) H_{k,1} + r_1 + 1 - \delta - \frac{1}{\beta} &= 0 \\
(\mu - \beta) H_{k,2} + r_2 + 1 - \delta - \frac{1}{\beta} &= 0 \\
\mu g_{e,1} - \chi \beta u_{c,1} w_1 &= 0 \\
\chi g_{e,2} - u_{c,2} w_2 &= 0,
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= (F(k_2, e_2) - \delta k_2) \\
c_2 &= (F(k_1, e_1) - \delta k_1).
\end{aligned}$$

Proposition 3 *If $\mu = \beta$ then the marginal products of capital and labor are at efficient levels, i.e., $r_1 = r_2 = r^*$ and $w_1 = w_2 = w^*$.*

The above result also implies that marginal products are independent of $H_1(k_1)$ and $H_2(k_2)$ and that the capital-labor ratios are efficient.

Proposition 4 *The Friedman rule implements the first best with Pareto weight $\theta = \frac{g_{e,1}}{\chi g_{e,2}}$.*

Proof. From the previous proposition, we have that the marginal products of capital and labor are at efficient levels. Thus, we are only left with checking two conditions.

The steady state of the planner's problem is

$$\begin{aligned}
\theta u_{c,2} w^* &= g_{e,1} \\
u_{c,1} w^* &= \theta g_{e,2}.
\end{aligned}$$

These two equations imply

$$u_{c,1}\ u_{c,2}\ {w^*}^2 = g_{e,1}\ g_{e,2}.$$

At the Friedman rule, the steady state of the monetary equilibrium solves the following conditions

$$\begin{aligned}
g_{e,1} &= \chi u_{c,1} w^* \\
\chi g_{e,2} &= u_{c,2} w^*,
\end{aligned}$$

which implies

$$u_{c,1}\ u_{c,2}\ {w^*}^2 = g_{e,1}\ g_{e,2}.$$

Thus, the planner's allocation and the monetary equilibrium at the Friedman rule imply the same steady state condition.

Now we verify that there exists a corresponding θ. From the monetary equilibrium conditions we have

$$u_{c,2}w^* = \chi \frac{g_{e,2}}{g_{e,1}}\, g_{e,1}.$$

Let $\theta = \frac{g_{e,1}}{\chi g_{e,2}}$ and so the above equation simplifies to

$$\theta u_{c,2}w^* = g_{e,1},$$

i.e., same as the first of the planner's condition.

The other condition from the monetary equilibrium can be written as

$$u_{c,1}w^* = \frac{g_{e,1}}{\chi g_{e,2}}\, g_{e,2}.$$

Apply the value for θ and we get

$$u_{c,1}w^* = \theta g_{e,2},$$

i.e., same as the second of the planner's condition. ■

Proposition 5 *For $H_2(k_2) = H_{k,2} = 0$ we have: (1) $r_2 = r^*$ and $w_2 = w^*$ in any monetary steady state; (2) $\chi = \frac{c1}{c2}$; and (3) the Friedman rule implements the first best with Pareto weight $\theta = 1$.*

7.4 Parametric example

To derive some more results, we now assume $u(c) = \log(c)$, $g(e) = \alpha e$, $H_i(k) = \eta_i k$, with $\eta_1 \geq 0, \eta_2 = 0$. From the proposition above, we know that the corresponding Pareto weight at the Friedman rule is $\theta = 1$. As a reference, the Pareto optimal levels of consumption for $\theta = 1$ are

$$c_1^* = c_2^* = \frac{w^*}{\alpha}.$$

Proposition 6 *In a monetary steady state, c_1, k_2 and e_2 are at efficient levels (for corresponding $\theta = 1$).*

Proof. From the steady state conditions of the monetary equilibrium we have $\chi g_{e,2} = u_{c,2}w_2$. Our assumptions imply $w_2 = w^*$, $\chi = c_1/c_2$ and thus we get $c_1 = c^*$. Since we also have k_2/e_2 at efficient levels, we get $c_1^* = k_2(F(1, e_2/k_2) - \delta)$ and thus $k_2 = k_2^*$ and $e_2 = e_2^*$. ■

This result implies that changing policy or institutional variables like μ and H_1, have no steady state effect on the consumption of agents type-1 nor the capital and effort of agents type-2.

Proposition 7 *If $\eta_1 = 0$ then a monetary steady state with higher μ features: (1) higher type-1 agent welfare; (2) lower type-2 agent welfare; (3) constant aggregate capital-output ratio; (4) lower aggregate capital stock.*

Proof. When $\eta_1 = 0$, $r_1 = r^*$ and $w_1 = w^*$. Thus, all marginal products are independent of μ and at efficient levels. From the steady state conditions of the monetary equilibrium we have $\mu g_{e,1} - \chi\beta u_{c,1} w^*$ which implies $c_2 = \frac{\beta w^*}{\alpha\mu}$. Thus, as μ increases, c_2 decreases. Since $e_2 = e_2^*$ for any μ, welfare in steady state for type-2 agents decreases with μ.

From the resource constraints we have $c_2 = k_1(F(1, e_1/k_1) - \delta)$. The capital-effort ratio is efficient and independent of μ. Thus, as c_2 decreases so does k_1, which in turn implies that e_1 decreases. Since $c_1 = c_1^*$ this implies that welfare in steady state for type-2 agents increases with μ.

Both capital-output ratios are at efficient levels and thus independent of μ. Thus, the aggregate capital-output ratio remains the same. However, since k_1 decreases with μ and k_2 remains constant, the aggregate capital stock decreases with μ. ■

Proposition 8 *If $\eta_1 > 0$ then a monetary steady state with higher μ features a higher aggregate capital-output ratio.*

Proof. With $\eta_1 > 0$, increasing μ lowers r_1 and increases w_1. Thus, k_1/e_1 increases. Since $H_2 = H_{k,2} = 0$, k_2/e_2 are at efficient levels and do not change with μ. Thus, the aggregate capital-output ratio in steady state increases with μ ■

Proposition 9 *Away from the Friedman rule, steady state welfare of type-2 agents increases as the marginal value of type-1 agents' collateral η_1 increases.*

Proof. For $\mu > \beta$, as η_1 increases, r_1 decreases and w_1 decreases. Since $c_2 = \frac{\beta w_1}{\alpha\mu}$, a higher w_1 implies a higher c_2. Since $e_2 = e^*$, we have that steady state welfare of type-2 agents increases. ■

Our numerical simulations show that in this case, the welfare of type-1 agents actually decreases.

8 References

1. Aruoba, S. Boragan and Randall Wright (2003). "Search, Money and Capital: a Neoclassical Dichotomy," *Journal of Money, Credit and Banking*, 35(6): 1086–1105.

2. Aruoba, S. Boragan, Christopher Waller and Randall Wright (2006). "Money and Capital," Manuscript.

3. Ferraris, Leo and Makoto Watanabe (2007). "Collateral Secured Loans in a Monetary Economy," Manuscript.

4. Hart, Oliver and John Moore (1994). "A Theory of Debt Based on the Inalienability of Human Capital," *The Quarterly Journal of Economics*, 109(4): 841–879.

5. Kehoe, Timothy J. and David K. Levine (1993). "Debt-Constrained Asset Markets," *Review of Economic Studies*, 60(4): 865–888.

6. Kiyotaki, Nobuhiro and John Moore (2002). "Evil is the Root of all Money," *The American Economic Review*, 92(2): 62–66.

7. Kocherlakota, Narayana (1998). "Money is Memory," *Journal of Economic Theory*, 81: 232–251.

8. Lagos, Ricardo and Guillaume Rocheteau (2004). "Money and Capital as Competing Media of Exchange," *Journal of Economic Theory* (forthcoming).

9. Lagos, Ricardo and Randall Wright (2005). "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113: 463–484.

10. Rocheteau, Guillaume and Randall Wright (2005). "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium," *Econometrica*, 73: 175–202.