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STABILITY IN A MODEL OF
STAGGERED-RESERVE ACCOUNTING

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Abstract

Critics of staggered-reserve accounting have used simple models to show that a disturbance to deposits with no change in total reserves sets in motion an undamped cycle in which deposits oscillate above and below the equilibrium implied by the total reserve target. In this paper a simple reduced-form model of the money-supply process is used to investigate the nature of the dynamic process implied by staggered-reserve accounting. The parameters in the model include the number of banking groups in the staggered regime, the reserve requirement, the response of banks to their own reserve position, and the response of banks to a deviation of the money supply from target.

Classical stability algorithms are used to find the range of parameters for which the model is stable. In this paper, the model is defined to be stable if the reduced-form difference equation for the money supply represents a converging process.

The results confirm the presence of a perpetual cycle found by others. This perpetual cycle depends on two special conditions: the first is that there are only two groups of banks in the staggering arrangement; the second is that banks ignore information about the money supply and Federal Reserve policy in making their asset portfolio decisions. When the model is extended to include more than two banking groups, or when banks are allowed to react to aggregate information, the money supply converges to the target level following a disturbance to equilibrium.

I. Introduction

In this paper we present a general model of the money-supply process with staggered-reserve accounting.¹ The fundamental building block is the multiplier relationship between total reserves and demand deposits. We abstract from the problems associated with lagged reserve accounting, the length of the settlement period, different types of deposits, differential reserve requirements, the demand for currency, the demand for excess reserves, and the uncontrollable factors affecting reserve supply. It is assumed that the Federal Reserve System sets the level of total reserves, closes the discount window, and allows no carryover.

The purpose of this model is to analyze assertions about the dynamic response of deposits to a disturbance of the money supply from the target level. This target level is assumed to be the equilibrium level in a more complete unspecified model of the money-supply process and the economy. Lindsey (1981) states that analysis done at the Federal Reserve Board on staggered-reserve accounting suggests that dynamic instabilities may be inherent in staggered accounting per se.² We find, as Trepeta and Lindsey (1979) have found, that under reasonable assumptions about banks' reactions to individual reserve shortages (or surpluses), random shocks to the money stock induce undamped cycles in the aggregate level of deposits. However, the failure of deposits to converge to the equilibrium level seems improbable because a cycle in deposits would tend to induce a cycle in the federal funds rate and imply a profit opportunity that would be easy for banks to exploit. In this paper we show that, even if banks ignore this profit opportunity, the dynamic instability described in the Trepeta-Lindsey paper is peculiar to a model with just two banking

groups. When the model is extended to include more than two banking groups, the dynamic instability disappears.

II. The Model

The model that is used to examine the assumptions about the institutional structure and economic behavior that are likely to produce dynamic instability in the money-supply process is shown in table 1. The banks are divided into n homogeneous groups, with one group settling in each of n successive periods. Required reserves are calculated on a contemporaneous basis over the n periods. Banks that are short of reserves on settlement day must borrow reserves from other banks. In the aggregate, if the members of a settling group are short of reserves, they must be net borrowers from the nonsettling group(s). It is assumed that deposits are split evenly among banking groups. The first equation is a behavioral equation in which the money supply changes in response to bank behavior. Changes in the money supply also depend on the nonbank public's behavior, but that behavior is ignored so that we may focus on the "cycling" phenomena reported by Laufenberg (1975) and Trepeta and Lindsey (1979).

The first equation describes aggregate behavior based on a model of a single bank's behavior. The equation contains two behavioral parameters, p and d ; p measures the reaction of an individual bank to a deviation between actual and required reserves. If the money supply goes above the target level, the demand for reserves will exceed the supply and each non-settling bank will borrow a proportion, p , of its deficit, causing interest rates to rise. As each bank adjusts the asset side of its balance sheet, deposits will tend to fall back to the target level.

Table 1

Model of a Staggered-Reserve Accounting Regime

$$(1.) \quad M_t = M_{t-1} - pn(RRN_{t-1} - ARN_{t-1}) \\ - d(M_{t-1} - 1/q \text{ TR}) - e_t.$$

$$(2.) \quad ARS_t = q/n \sum_{i=0}^{n-1} M_{t-i} - \sum_{i=1}^{n-1} ARN_{t-i}.$$

$$(3.) \quad ARN_t = (TR - ARS_t)/(n - 1).$$

M = the money supply

TK = total reserves

RRN = required reserves of a typical non-settling group

ARN = actual reserves of a typical non-settling group

ARS = actual reserves of the settling banks

e = exogenous shocks

p = proportion of reserve imbalance that banks try to make up in one period

d = adjustment by banks to a deviation of the money supply from target

q = required reserve ratio

n = number of banking groups and number of weeks in the reserve accounting period

Since total reserves are fixed, banks will not be able to achieve the portfolio mix desired at the initial level of interest rates. Therefore, prices of other assets will change until the individual banks are satisfied to hold available reserves. The settling banks will have to borrow any deficit (and are assumed to lend all excess reserves) on settlement day. The greater the size of the reserve imbalance, the more interest rates will have to change. The actions of the settling banks will affect the money supply in the same way as the actions of the non-settling banks. Therefore, the reserve deficit or surplus of a typical non-settling banking group is multiplied by the number of banking groups.³ If $p = 1$, then the total deposit effect of the individual bank reactions to their own reserve positions will be equal to n times the difference between the required and actual reserves of a non-settling banking group. If the reserve requirement is less than 100 percent, this effect is less than the amount by which total deposits would have to change to return the money supply to the target level after one week.

The other behavioral parameter is d . It measures aggregate bank reaction to a deviation of the money supply from the target. Equation 1 incorporates reaction to two types of information. The first is internal and represents an individual bank's reaction to its own portfolio position. The second is the reaction to aggregate information available to all banks. In addition, this equation includes a disturbance term that represents shocks that originate from outside this model.

Equation 2 is also a behavioral equation. It is based on the assumption that the settling banks will never hold excess reserves and that they will always meet their reserve requirements.

Equation 3 is a definition used to calculate actual reserves of a typical group of non-settling banks. The discount window is closed, and there is no carryover so that actual reserves of the settling banks will equal required reserves. Actual reserves outside the settling banks are assumed to be divided evenly among the non-settling banks.

The general solution for the dynamic path of the money supply in response to exogenous shocks is given below. Noting that $RRN_t = q/n M_t$ and substituting 2 into 3, the model reduces to equations 1 and 3':

$$(1) \quad M_t = M_{t-1} - pn(q/n M_{t-1} - ARN_{t-1}) - d(M_{t-1} - 1/q TR) - e_t.$$

$$(3') \quad ARN_t = \frac{1}{n-1} \left(TR - q/n \sum_{i=0}^{n-1} M_{t-i} + \sum_{i=1}^{n-1} ARN_{t-i} \right).$$

Using lag operators and substituting equation 3' into 1, we get

$$(4) \quad M_t = M_{t-1} - pn \left[q/n M_{t-1} - \frac{TR - q/n \sum_{i=1}^n M_{t-i}}{n-1 - \sum_{i=1}^{n-1} B^i} \right] - d \left(M_{t-1} - 1/q TR \right) - e_t.$$

This equation shows that the time path of M_t will be determined by a combination of the response to individual portfolio imbalances and the response to aggregate data. Solution of the general case requires the solution of an n^{th} order polynomial with parameters p , q , and d .

III. Stability Conditions When There Is No Reaction to Aggregate Data

The problem is simplified somewhat if we assume that there is no response to aggregate data. Therefore, we begin by analyzing the case in which $d = 0$. Analytical solutions for the range of pq for which this model is stable can be derived for $n = 2, 3, \text{ or } 4$.

The difference equation derived from equation 4 under the assumption that $d = 0$ is given by

$$(5) \quad M_t = 1/q \text{ TR} - \frac{\left(n - 1 - \sum_{i=1}^{n-1} B^i \right) e_t}{n - 1 - (n-npq)B + B^n}$$

This process will converge if all the roots of the polynomial, $n - 1 - (n - npq)B + B^n$, lie outside the unit circle. If any root lies on or within the unit circle, the process will not converge.

For the case in which there are two banking groups, the polynomial in the denominator of equation 5 is

$$1 - 2(1 - pq)B + B^2.$$

The roots of the polynomial are given by

$$1 - pq \pm ((pq)^2 - 2pq)^{\frac{1}{2}}.$$

For pq equal to 0, both roots equal 1; for pq equal to 2, both roots equal -1. For pq inside the range 0 to 2, the roots are complex; the distance of the roots from the origin is given by

$$[(1 - pq)^2 + 2pq - (pq)^2]^{\frac{1}{2}} = 1.$$

In this special case of two banking groups with $d = 0$, $0 \leq pq \leq 2$, both roots lie on the unit circle, and the path for the money supply following an exogenous shock is an undamped cycle.

A priori, one would expect pq to be in this range. The reserve requirement, q , is in the neighborhood of 10 percent. The parameter, p , describing the banks' reactions to reserve imbalances should be in the neighborhood of unity. Since reserves yield no explicit return, excess reserves are held only for precautionary reasons, and there is little reason for a bank to stockpile reserves. Negative values of p are unlikely if banks behave in a rational manner.

If the product pq is greater than 2 or less than 0, both roots will be real. One root will be less than 1 in absolute value; the other root will be greater than 1 in absolute value, and the dynamic process described in equation 5 explodes. The economic interpretation is straightforward. If $pq < 0$, then banks are persistently lending at an interest rate that is below the expected interest rate in the next period, or they are persistently borrowing at an interest rate that is above the expected interest rate in the next period. If pq is greater than 2, and q is in the neighborhood of 10 percent, then banks are greatly overreacting to any reserve imbalance.

To analyze the cases when n is greater than 2, note that the polynomial in equation 6 has roots that are the inverse of the roots in equation 7.

$$(6) \quad f(B) = n-1 - n(1 - pq)B + B^n.$$

$$(7) \quad g(B) = 1 - n(1 - pq)B^{n-1} + (n-1)B^n.$$

Duffin (1969) presented an algorithm for testing whether all the roots of a polynomial lie inside the unit circle. If the roots of $g(B)$ lie inside the unit circle, then the roots of $f(B)$ lie outside the unit circle, and equation 5 represents a stable difference equation. The details of this algorithm are presented in appendix A with the solutions for the cases $n = 3$ and $n = 4$. Equation 5 is stable for the following values of p and q :

Case 1: $n = 3$,
 $0 < pq < 4/3$.

Case 2: $n = 4$,
 $0 < pq < 2$.

For the general case of n banking groups, analytic calculation of bounds on pq for convergence proved to be intractable. However, following computer solutions of the model for cases n equals 5 through 30 and a large variety of values for pq , we hypothesize that the dynamic process described in equation 5 converges for the following values of pq :

Case 1: $n > 2$ and odd,

$$0 < pq < 2 \left(\frac{n-1}{n} \right).$$

Case 2: $n > 2$ and even,

$$0 < pq < 2$$

IV. Stability Conditions with a Reaction to Aggregate Information

In this section we relax the assumption that $d = 0$. The difference equation derived from equation 4 in the general case is given by

$$(8) \quad M_t = 1/q \mathbb{R} - \frac{\left(n - 1 - \sum_{i=1}^{n-1} B^i \right) e_t}{n - 1 - n(1-pq-d)B + B^n - d \sum_{i=1}^n B^i}$$

When $n = 2$, equation 8 is stable if and only if both roots of the polynomial

$$(9) \quad 1 - [2(1 - pq) - d]B + (1 - d)B^2$$

lie outside the unit circle. For the general case the procedure developed by Duffin and used above is intractable for $n > 2$. Instead, we use a method developed by Wise (1956) to derive stability conditions. For $n = 2$, either method yields the following conditions (see appendix B for details):

(i) $0 < d < 2,$

(ii) $0 < pq < 2 - d.$

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It is important to note that while the system with two banking groups did not converge for any values of pq when $d = 0$, it converges for all likely values of p and q when d lies between 0 and 2. The great attention attached to the weekly release of aggregate information on the money supply suggests that d is likely to be greater than 0. While the actual value of d can only be estimated, we know that banks and the public react to aggregate information in today's market. Unless one would predict a drastic change in behavior following the adoption of staggered-reserve accounting, d is likely to fall in the stable range.

Appendix B also describes the algorithm used to compute the solutions to equation 8 for higher values of n . Numeric solutions up to the case $n = 30$ indicate the following stability conditions when $d > 0$.

Case 1: $n > 2$ and odd,

$$0 < pq < \frac{n-1}{n}(2 - d).$$

Case 2: $n > 2$ and even,

$$0 < pq < 2 - d.$$

For $d < 0$ and $n > 2$, there is a stable region for some values of pq . The cases for $n = 3, 4,$ and 5 are shown in appendix B and plotted in figure B.2.

V. Conclusion

The dynamic process described by equation 5 includes no systematic reaction in the market to aggregate information. When there are only two banking groups with staggered reserve-settlement periods, we find that this process represents an undamped, unexplosive cycle for likely values of p , the bank portfolio response, and q , the reserve requirement. When the number of banking groups is increased to 3 or 4, and for likely values of p and q , the money supply converges in a damped cycle to the target level.

The dynamic process described in equation 8 includes a market response to aggregate information. If market participants respond to a deviation of the money supply from target by changing interest rates in the direction of the deviation, then the dynamic process described in equation 8 is likely to converge even when there are only two banking groups.

Footnotes

1. The proposal to stagger reserve accounting was first made by Cox and Leach (1964); also see comments following the article by Sternlight (1964). This proposal was resurrected by the Morgan Guaranty Company (1981) and discussed by Gavin (1982).
2. These results come from the analysis of a model constructed by Trepeta; see the appendix to Trepeta and Lindsey (1979). The fundamental difference between the model presented here and Trepeta and Lindsey's is that we generalize to include more than two banking groups, and we add a behavioral parameter to capture reaction to aggregate information. Laufenberg (1975) first noted that the institutional structure of staggered-reserve accounting implied the possibility of dynamic instabilities.
3. Robert Avery, Federal Reserve Board staff, has pointed out that when $n = 2$, and the model is changed so that

$$M_t = M_{t-1} - p(n-1) (RRN_{t-1} - ARN_{t-1}) \\ - d(M_{t-1} + 1/q TR) - e_t,$$

the range of values of pq for which the model is stable is reduced by a factor of one-half. Changing the model in this way does not change any of our qualitative results.

Appendix A

Stability in the Model with $d = 0$

A Test for Schur Polynomials from Duffin (1969)

A Schur polynomial is defined as a polynomial for which all the roots lie inside the unit circle.

Algorithm. Let $g_n(w)$ be the polynomial

$$g_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n,$$

where

$a_0 \neq 0$, $a_n \neq 0$. Let $g_{n-1}(x)$ be the reduced polynomial

$$g_{n-1}(x) = (a_n a_1 - a_0 a_{n-1}) + (a_n a_2 - a_0 a_{n-2})x + \dots + (a_n a_n - a_0 a_0)x^{n-1}$$

of degree $n - 1$. Then $g_n(x)$ is a Schur polynomial if and only if

(i) $|a_0| < |a_n|$, and

(ii) $g_{n-1}(x)$ is a Schur polynomial.

This procedure is iterated n times to get the parameter domain for which $g_n(x)$ is a Schur polynomial.

Case: $n = 3$

$$g_3(B) = 1 - 3(1 - pq)B^2 + 2B^3.$$

(i) $|1| < |2|$,

(ii) $g_2(B) = 3(1 - pq) - 6(1 - pq)B + 3B^2$,

(i)' $|3(1 - pq)| < 3 \iff$

$|1 - pq| < 1 \iff 0 < pq < 2$,

(ii)' $g_1(B) = -18(1 - pq) + 18(1 - pq)^2 + (9 - 9(1 - pq)^2)B$,

$$\begin{aligned} \text{(ii)'' } 2|-(1 - pq) + (1 - pq)^2| &< |1 - (1 - pq)^2| \iff \\ 2|(1 - pq) (-1 + 1 - pq)| &< |1 - (1 - pq)^2| \iff \\ 2|1 - pq| |-pq| &< |1 - (1 - pq)^2|. \end{aligned}$$

From above $pq > 0$, and $|1 - pq| < 1$ implies $1 - (1 - pq)^2 > 0$.

Thus, (ii)'' is equivalent to

$$2pq |1 - pq| < 1 - (1 - pq)^2.$$

When $1 - pq > 0$

$$\begin{aligned} 2pq (1 - pq) &< 1 - (1 - 2pq + (pq)^2) \iff \\ 2pq - 2(pq)^2 &< 2pq - (pq)^2 \iff \\ -2(pq)^2 &< -(pq)^2 \text{ which is always true.} \end{aligned}$$

When $1 - pq < 0$

$$\begin{aligned} 2pq (pq - 1) &< 2pq - (pq)^2 \iff \\ 3(pq)^2 - 4pq &< 0 \iff \\ 3pq - 4 &< 0 \iff \\ pq &< 4/3. \end{aligned}$$

Therefore, when $n = 3$ and $0 < pq < 4/3$, equation 5 represents a stable difference equation.

Case: $n = 4$

$$g_4(B) = 1 - 4(1 - pq)B^3 + 3B^4.$$

$$\text{(i) } |1| < |3|,$$

$$\text{(ii) } g_3(B) = 4(1 - pq) - 12(1 - pq)B^2 + 8B^3,$$

$$(i)' \quad |4(1 - pq)| < 8 \Leftrightarrow -1 < pq < 3.$$

$$(ii)' \quad g_2(B) = 48(1 - pq)^2 - 96(1 - pq)B + (64 - 16(1 - pq)^2)B^2.$$

$$(i)'' \quad |48(1 - pq)^2| < |64 - 16(1 - pq)^2|.$$

Both $(1 - pq)^2$ and $4 - (1 - pq)^2$ are greater than 0 on the range $-1 < pq < 3$. Therefore, $(i)'' \Leftrightarrow$

$$3(1 - pq)^2 < 4 - (1 - pq)^2 \Leftrightarrow$$

$$(1 - pq)^2 < 1,$$

$$0 < pq < 2.$$

$$(ii)'' \quad g_1(B) = [-(64 - 16(1 - pq)^2)(96(1 - pq)) + (48(1 - pq)^2)(96(1 - pq))] + [(64 - 16(1 - pq)^2)^2 - (48(1 - pq)^2)^2]B.$$

$$|-(64 - 16(1 - pq)^2)(96(1 - pq)) + 48(1 - pq)^2 96(1 - pq)| <$$

$$|(64 - 16(1 - pq)^2)^2 - (48(1 - pq)^2)^2| \Leftrightarrow$$

$$A.1 \quad |3(1 - pq) - 3(1 - pq)^3| < |2 - (1 - pq)^2 - (1 - pq)^4|$$

for $0 < pq < 1$

$$3(1 - pq) - 3(1 - pq)^3 > 0 \quad \text{and}$$

$$2 - (1 - pq)^2 - (1 - pq)^4 > 0.$$

$$\text{Therefore, } A.1 \Leftrightarrow 2 - 3(1 - pq) - (1 - pq)^2 + 3(1 - pq)^3 - (1 - pq)^4 = f(1 - pq) > 0.$$

$$f(1 - pq) = 0 \quad \text{when } pq = 0 \\ = 2 \quad \text{when } pq = 1$$

To show that $f(1 - pq) > 0$ for all $0 < pq < 1$ note that

$$f'(1 - pq) = 3 + 2(1 - pq) - 9(1 - pq)^2 + 4(1 - pq)^3$$

$$f'(1 - pq) = 0 \text{ when } pq = -.693, 0, \text{ and } 1.443.$$

$$f'(1 - .5) = 2.25.$$

Therefore, $f'(1 - pq) > 0$ for $0 < pq < 1$, and $f(1 - pq) > 0$ for $0 < pq < 1$.

For $1 < pq < 2$

$$3(1 - pq)^3 - 3(1 - pq) > 0 \quad \text{and}$$

$$2 - (1 - pq)^2 - (1 - pq)^4 > 0.$$

$$\begin{aligned} \text{Therefore, } A.1 &\Leftrightarrow 2 + 3(1 - pq) - (1 - pq)^2 - 3(1 - pq)^3 - (1 - pq)^4 \\ &= h(1 - pq) > 0. \end{aligned}$$

$$\begin{aligned} h(1 - pq) &= 2 \text{ when } pq = 1, \\ &= 0 \text{ when } pq = 2. \end{aligned}$$

To show that $h(1 - pq) > 0$ for all $1 < pq < 2$ note that

$$h'(1 - pq) = -3 + 2(1 - pq) + 9(1 - pq)^2 + 4(1 - pq)^3.$$

$$h'(1 - pq) = 0 \text{ when } pq = .557, 2, \text{ and } 2.693.$$

$$h'(1 - 1.5) = -2.25.$$

Therefore, $h'(1 - pq) < 0$ for $1 < pq < 2$, and $h(1 - pq) > 0$ for $1 < pq < 2$.

Therefore, equation 5 represents a stable difference equation when $n = 4$ and $0 < pq < 2$.

For the general case of n banking groups, analytic calculation of bounds on pq for convergence proved to be intractable. However, following computer solutions of the model for cases n equals 5 through 30 and a large variety of values for pq , we hypothesize that the dynamic process described in equation 5 converges for the following values of pq .

Case 1: $n > 2$ and odd,
 $0 < pq < 2 \left(\frac{n-1}{n}\right)$.

Case 2: $n > 2$ and even,
 $0 < pq < 2$.

Appendix B

Stability of the Model in the General Case

The dynamic process described in equation 8 will be stable if the roots of

$$B.1 \quad B^{n-1} - n(1 - pq - d)B + B^n - d \sum_{i=1}^n B^i = 0$$

lie outside the unit circle. Equivalently, the system will be stable if the roots of

$$B.2 \quad B^n - \frac{n(1 - pq - d)}{n-1} B^{n-1} + \frac{1}{n-1} - \frac{d}{n-1} \sum_{i=0}^{n-1} B^i = 0$$

lie inside the unit circle.

Wise (1956) presents a method of transforming equations of the type B.2 such that the conditions on roots lying inside the unit circle are equivalent to the real part of the roots of

$$B.3 \quad p_0 + p_1 y + \dots + p_n y^n = 0,$$

being less than zero. In B.3 y and p_r are defined as

$$y = \frac{B + 1}{B - 1} \quad \text{and}$$

$$p_r = \sum_{j=0}^n \alpha_j C_{rj}.$$

C_{rj} is the coefficient of y^r in the expansion of

$$(y + 1)^{n-j} (y - 1)^j,$$

$$\alpha_0 = 1,$$

$$\alpha_1 = - \frac{n(1 - pq) - (n - 1)d}{n-1},$$

$$\alpha_2 = \dots = \alpha_{n-1} = - \frac{d}{n-1}, \quad \text{and}$$

$$\alpha_n = \frac{1 - d}{n - 1}.$$

From this point, we can use the Routh theorem (Chiang 1974, p. 546), which states that a polynomial of the form (with a_0 assumed greater than 0)

$$B.4 \quad a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$$

has the real parts of all its roots negative **if and only if** the first n of the following sequence of determinants

$$\begin{vmatrix} a_1 \end{vmatrix}; \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}; \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}; \dots$$

are all positive. In applying this theorem, it should be remembered that

$$|a_1| \equiv a_1 \text{ and that we set}$$

$$a_m = 0 \text{ for } m > n.$$

In the present problem, we have

$$a_i = p_{n-i} \text{ for } i = 0, 1, \dots, n.$$

Thus, the conditions for stability of the system can theoretically be derived from Wise's results and the Routh theorem. However, in practice the analytical solution is extremely complicated. Therefore, we present the analytical solutions for $n=2$ and 3. Numeric solutions are given for $n > 3$.

Case: $n=2$

$$\alpha_1 = -[2(1-pq)-d] \text{ and}$$

$$\alpha_2 = 1 - d.$$

From Wise (1956):

$$p_0 = 1 - \alpha_1 + \alpha_2 = 2[2 - pq - d],$$

$$p_1 = 2 - 2\alpha_2 = 2d \text{ and}$$

$$p_2 = 1 + \alpha_1 + \alpha_2 = 2pq.$$

Thus, $a_0 = p_2 = 2pq$ is positive if $pq > 0$. Otherwise, all signs of the p_i 's must be reversed. If $pq > 0$, then the conditions of the Routh theorem are

$$|a_1| = p_1 > 0 \text{ or equivalently, } d > 0 \text{ and}$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} p_1 & 0 \\ p_2 & p_0 \end{vmatrix} = p_1 p_0 > 0 \text{ or}$$

$$2d[2(2 - pq - d)] > 0. \text{ Since } d > 0, \text{ we get}$$

$$2 - pq - d > 0 \text{ or}$$

$$pq < 2 - d.$$

If $pq < 0$, then we have to reverse signs before applying the Routh theorem.

Conditions are then

$$|a_1| = -p_1 = -2d > 0 \text{ or } d < 0.$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} -p_1 & 0 \\ -p_2 & -p_0 \end{vmatrix} = p_1 p_0 > 0 \iff$$

$$2d[2(2 - pq - d)] > 0. \text{ Again, since } d < 0$$

$$2 - pq - d < 0 \iff$$

$$pq > 2 - d > 0.$$

This last inequality contradicts $pq < 0$. Thus, there are no stable solutions for which $pq < 0$. The stable region is given by

$$0 < pq < 2 - d \text{ and}$$

$$d > 0.$$

It is shown in figure B.1.

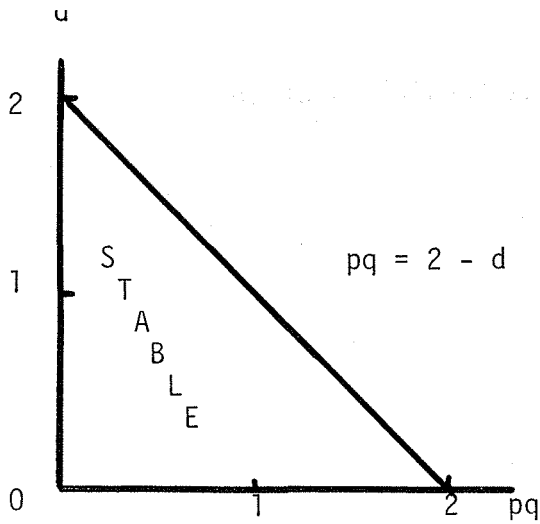


Figure B.1 Stable Parameter Domain

Case: $n = 3$

$$\alpha_1 = - \frac{[3(1 - pq) - 2d]}{2},$$

$$\alpha_2 = - \frac{d}{2}, \text{ and}$$

$$\alpha_3 = \frac{1 - d}{2}$$

From Wise (1956):

$$p_0 = 1 - \alpha_1 + \alpha_2 - \alpha_3 = \frac{4 - 3pq - 2d}{2},$$

$$p_1 = 3 - \alpha_1 - \alpha_2 + 3\alpha_3 = \frac{12 - 3pq - 4d}{2}$$

$$p_2 = 3 + \alpha_1 - \alpha_2 - 3\alpha_3 = \frac{3pq + 6d}{2}, \text{ and}$$

$$p_3 = 1 + \alpha_1 + \alpha_2 + \alpha_3 = \frac{3pq}{2}.$$

Again, this case divides into two subcases, depending on whether p_3 is positive or negative.

If $p_3 > 0$ or equivalently, $pq > 0$,

$$|a_1| = p_2 > 0 \iff pq > -2d.$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} p_2 & p_0 \\ p_3 & p_1 \end{vmatrix} = p_2 p_1 - p_0 p_3 > 0$$

$$\Leftrightarrow \frac{(3pq + 6d)}{2} \frac{(12 - 3pq - 4d)}{2} - \frac{(4 - 3pq - 2d)}{2} \frac{(3pq)}{2} > 0$$

$$\Leftrightarrow pq(1 - d) > d^2 - 3d \Leftrightarrow$$

$$\text{B.5 } pq < d \frac{(d - 3)}{1 - d} \text{ for } d > 1 \text{ and}$$

$$pq > d \frac{(d - 3)}{1 - d} \text{ for } d < 1.$$

There is no constraint from this condition on pq when $a = 0$.

$$\begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} p_2 & p_0 & 0 \\ p_3 & p_1 & 0 \\ 0 & p_2 & p_0 \end{vmatrix} = p_2 p_1 p_0 - p_0^2 p_3 > 0.$$

If $p_0 > 0$, we get the same conditions as in B.5 plus $p_0 > 0$

$$\Leftrightarrow 4 - 3pq - 2d > 0 \text{ or}$$

$$\text{B.6 } pq < \frac{4 - 2d}{3}.$$

If $p_0 < 0$, we get the reverse of conditions given in B.5. Because we cannot have both the conditions in B.5 and their inverses, there is no stable solution when $p_0 < 0$.

If $p_3 < 0$ or equivalently $pq < 0$, we have

$$|a_1| = -p_2 > 0 \Leftrightarrow pq < -2d,$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} -p_2 & -p_0 \\ -p_3 & -p_1 \end{vmatrix} = p_2 p_1 - p_0 p_3 > 0$$

(which gives the same condition as B.5) and

$$\begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} -p_2 & -p_0 & 0 \\ -p_3 & -p_1 & 0 \\ 0 & -p_2 & -p_0 \end{vmatrix} = -p_2 p_1 p_0 + p_0^2 p_3 > 0.$$

If $p_0 > 0$, we get the reverse of conditions in B.5. Thus, there is no stable region with $p_0 > 0$ and $p_3 < 0$. For $p_0 < 0$, we obtain the same conditions as B.5 and

$$4 - 3pq - 2d < 0$$

or

$$pq > \frac{4 - 2d}{3}.$$

Because it is not possible to have both $pq < -2d$ and $pq > (4 - 2d)/3$, there are no stable conditions for $pq < 0$.

In summary, the stable region for $n = 3$ is given by the set of conditions obtained from those given above by determining which ones are binding conditions. They are

$$pq > 0, \quad pq < -2d, \quad \text{and} \quad pq > \frac{4 - 2d}{3}$$

$$pq < \frac{4 - 2d}{3}, \quad \text{and}$$

$$pq > d \frac{(d - 3)}{1 - d} \quad \text{for } d < 0.$$

Theoretically, it would be possible to solve for the stability regions by using the above method for $n > 3$. However, the application is not practical because of the complicated expressions involved. Instead, we have developed computer solutions for n up to 30 using the above method and an algorithm developed by Duffin (1969) to calculate the C_{rj} 's. From these solutions, we make the following hypothesis:

(1) For $d > 0$, the stability region is given by

$$pq < \frac{n-1}{n} (2-d) \text{ for } n \text{ odd,}$$

$$pq < 2-d \text{ for } n \text{ even.}$$

(2) For $d < 0$, there is a stability region for which we have not been able to determine general formulas for boundaries except partially for the case where n is odd. In this case, the upper boundary appears to be $pq < \frac{n-1}{n} (2-d)$. Figure B.2 illustrates stability regions for cases in which $n = 3, 4,$ and 5 .

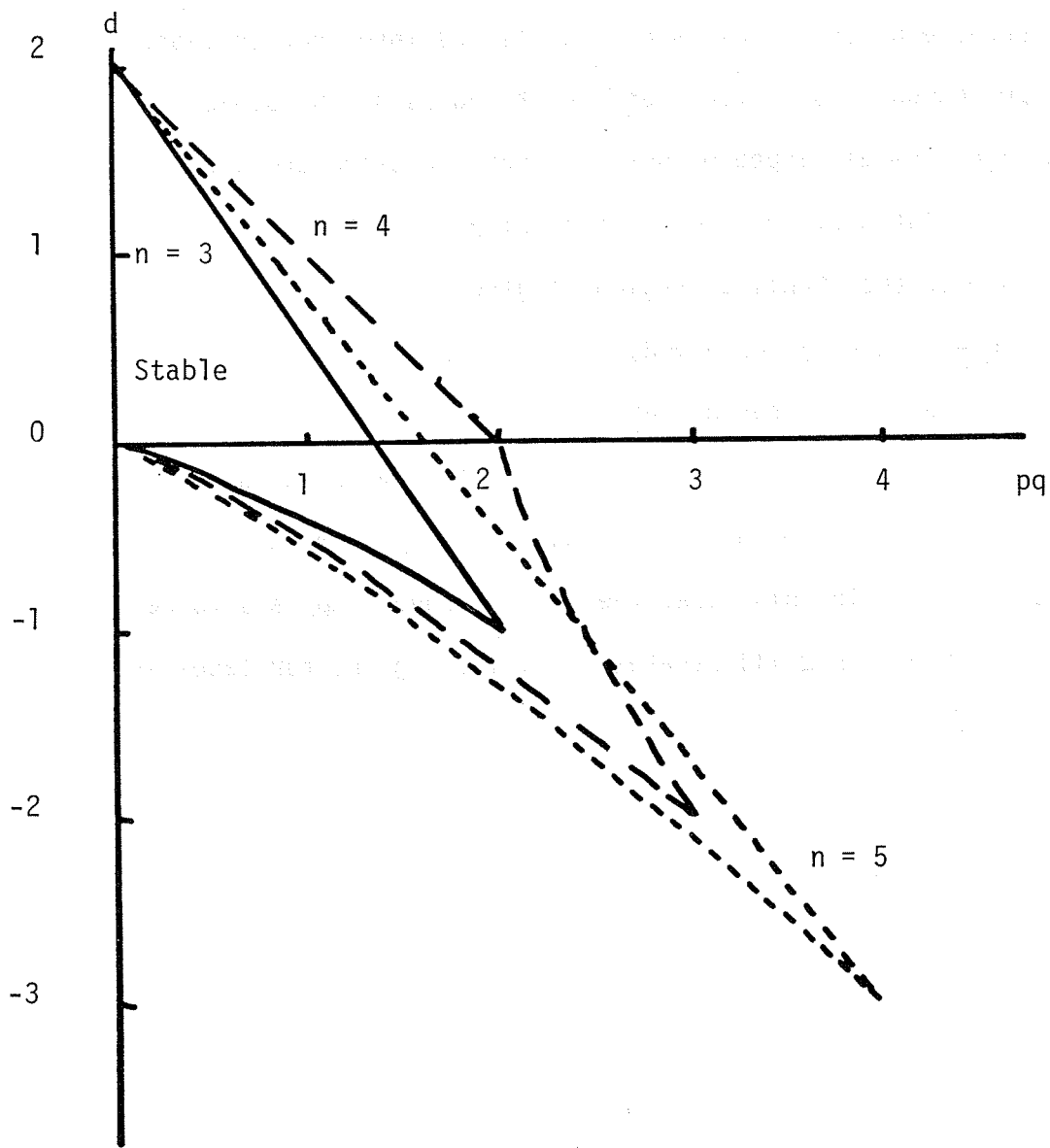


Figure B.2. Stable Parameter Domains

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